Exact Solutions for Some Reaction Diffusion Systems with Nonlinear Reaction Polynomial Terms

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Abstract

We introduce a method to find exact traveling wave solutions for reaction diffusion systems of equations with nonlinear reaction polynomial terms. We will apply the method on an important kinds of reaction diffusion systems, such as the model for the spatial spread of an epidemic, predator-prey two component telegraph model.

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1 Introduction

Several models that appears in many branches of sciences are described by systems of partial differential equations [1-4]. Rosu and Cornejo [5,6] have proved that for some nonlinear second order ordinary differential equations it is a very simple task to find one particular solution once the nonlinear equation is factorized with the use of two first order differential operators. The paper is organized as follows: In section 2, we introduce the factorization scheme for system of two second ordinary differential equations with nonlinear reaction polynomial term, and this leads to an easy finding of analytical solutions. In
section 3, we use the factorization method that presented in section 2 to find an explicit particular solutions for three models: Model of reaction diffusion system of two equations, simple model for the spatial spread of an epidemic, and Predator-Prey two component telegraph model [7].

2 Factorization procedure for nonlinear ordinary second order differential equations

Factorization of second-order linear differential equations is a well established technique to find solutions in an algebraic manner. Rosu and Cornejo [5-7] has been fined one particular solution once the nonlinear equation is factorized with the use of two first order differential operators. They use the method for equations of types:

\[ u'' + \gamma u' + f(u) = 0 \] (1)

and

\[ u'' + g(u)u' + f(u) = 0 \] (2)

We concentrate our work in this paper to a technique of factorization for systems of differential equation, namely:

\[ u'' + g_1(u,v)u' + f_1(u,v) = 0, \] (3a)

\[ v'' + g_2(u,v)v' + f_2(u,v) = 0, \] (3b)

where ' means the derivative \( D = \frac{d}{dz} \), \( g_i(u,v) \) and \( f_i(u,v) \), \( i = 1, 2 \) are polynomials in \( u, v \).

Now, system (3) can be written in the operator form as:

\[ [D^2 + g_1(u,v)D + \frac{f_1(u,v)}{u}]u = 0, \] (4a)

\[ [D^2 + g_2(u,v)D + \frac{f_2(u,v)}{v}]v = 0. \] (4b)

To make a factorization for system (4) \( f_1(u,v) \) and \( f_2(u,v) \) must be take the following form

\[ f_1(u,v) = u \ h_1(u,v), \quad f_2(u,v) = v \ h_2(u,v). \] (5)

Now equations (4) can be factorized as:

\[ [D - \Psi_{12}(u,v)][D - \Psi_{11}(u,v)]u = 0, \] (6a)

\[ [D - \Psi_{21}(u,v)][D - \Psi_{22}(u,v)]v = 0, \] (6b)

which leads to the equations
\[
\begin{align*}
\varepsilon'' - \left( \Psi_{12} + \Psi_{11} + \frac{\partial \Psi_{11}}{\partial u} u \right) u' + \Psi_{12} \Psi_{11} u &= 0, \\
\varphi'' - \left( \Psi_{21} + \Psi_{22} + \frac{\partial \Psi_{22}}{\partial v} v \right) v' + \Psi_{21} \Psi_{22} v &= 0.
\end{align*}
\]

Comparing (7) and (3) we obtain
\[
\begin{align*}
g_1(u, v) &= - \left( \Psi_{12} + \Psi_{11} + \frac{\partial \Psi_{11}}{\partial u} u \right), \\
g_2(u, v) &= - \left( \Psi_{21} + \Psi_{22} + \frac{\partial \Psi_{22}}{\partial v} v \right).
\end{align*}
\]

and
\[
\begin{align*}
f_1(u, v) &= \Psi_{12} \Psi_{11} u, \\
f_2(u, v) &= \Psi_{21} \Psi_{22} v.
\end{align*}
\]

If \( f_1(u, v), f_2(u, v) \) are polynomial functions, then \( g_1(u, v), g_2(u, v) \) will have the same order as the bigger of the factorizing functions, \( \Psi_{12}, \Psi_{11}, \Psi_{21} \) and \( \Psi_{22} \) and will also be a function of the constant parameters. Then system (3) transformed to four possible systems of first order differential equations
\[
\begin{align*}
u' - \Psi_{11}(u, v) u &= 0, \\
v' - \Psi_{22}(u, v) v &= 0, \\
u' - \Psi_{12}(u, v) u &= 0, \\
v' - \Psi_{21}(u, v) v &= 0,
\end{align*}
\]

and by a certain choice for \( \Psi_{11}, \Psi_{22} \) we can obtain a particular solutions for the system (3).

## 3 Applications

### 3.1 Model of reaction diffusion system of two equations

Consider the following reaction diffusion system
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + f_1(u, v), \quad f_1(u, v) = u (1 - u)(1 - v), \\
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + f_2(u, v), \quad f_2(u, v) = v (1 - v)(1 - u).
\end{align*}
\]
Using the coordinate transformation \( z = x - ct \) (\( c \) is the propagation speed), then we have

\[
\begin{align*}
  u'' + g_1(u, v)u' + f_1(u, v) &= 0, \\
  v'' + g_1(u, v)v' + f_2(u, v) &= 0.
\end{align*}
\]

(12a) (12b)

where

\[
\begin{align*}
  g_1(u, v) &= c, & f_1(u, v) &= u(1 - u)(1 - v), \\
  g_2(u, v) &= c, & f_2(u, v) &= u(1 - u)(1 - v).
\end{align*}
\]

(13a) (13b)

The factorization of (12) leads to

\[
\begin{align*}
  [D - \Psi_{12}(u, v)] [D - \Psi_{11}(u, v)] u &= 0, \\
  [D - \Psi_{21}(u, v)] [D - \Psi_{22}(u, v)] v &= 0.
\end{align*}
\]

(14a) (14b)

Now, choosing \( \Psi_{ij} \) such that

\[
\begin{align*}
  \Psi_{12} &= \frac{1}{k_1}(1 - v), & \Psi_{11} &= k_1(1 - u), \\
  \Psi_{21} &= \frac{1}{k_2}(1 - u), & \Psi_{22} &= k_2(1 - v),
\end{align*}
\]

(15a) (15b)

where \( k_1, k_2 \) are arbitrary constants must be determined and then

\[
\begin{align*}
  u' - k_1 u(1 - u) &= 0, & v' - k_2 v(1 - v) &= 0.
\end{align*}
\]

The first conditions in equation (8) lead to \( k_1 = \frac{-c + \sqrt{c^2 + 1}}{2}, k_2 = \frac{-c - \sqrt{c^2 + 1}}{2}. \)

\[
\begin{align*}
  u^+(z) &= \left[ \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{k_1}{2} (z - z_0) \right) \right], & \text{or } u^-(z) &= \left[ \frac{1}{2} - \frac{1}{2} \coth \left( \frac{k_1}{2} (z - z_0) \right) \right].
\end{align*}
\]

(16)

and

\[
\begin{align*}
  v^+(z) &= \left[ \frac{1}{2} - \frac{1}{2} \tanh \left( \frac{k_2}{2} (z - z_0) \right) \right], & \text{or } v^-(z) &= \left[ \frac{1}{2} - \frac{1}{2} \coth \left( \frac{k_2}{2} (z - z_0) \right) \right].
\end{align*}
\]

(17)
3.2 Simple model for the spatial spread of an epidemic

In this subsection we will obtain particular solutions for simple model for the spatial spread of an epidemic \cite{8,9} by using the method presented in Sec. 2. The model for the spatial spread of an epidemic is given as:

\[
\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2} + I(S - \lambda), \quad \frac{\partial S}{\partial t} = \frac{\partial^2 S}{\partial x^2} - IS, \tag{18}
\]

where \(I, S\) are the population density and \(\lambda\) is a constant (measure of the mortality rate as compared with the constant rate).

Using the coordinate transformation \(z = x - ct\) in equation (18) we obtain the following nonlinear ordinary differential system

\[
I'' + g_1(I', S)I' + f_1(I, S) = 0, \tag{19a}
\]

\[
S'' + g_2(I, S)S' + f_2(I, S) = 0, \tag{19b}
\]

where

\[
g_1(I, S) = c, \quad g_2(I, S) = c, \quad f_1(I, S) = I(S - \lambda) \quad \text{and} \quad f_2(I, S) = -IS. \tag{20}
\]

System (19) can be written using operator notations in the form:

\[
[D^2 + g_1(I', S)D + \frac{f_1(I, S)}{I}]I = 0, \quad [D^2 + g_2(I', S)D + \frac{f_2(I, S)}{S}]S = 0, \tag{21}
\]

Factorization of (21) leads to

\[
[D - \Psi_{12}(I, S)][D - \Psi_{11}(I, S)]I = 0, \tag{22a}
\]

\[
[D - \Psi_{21}(I, S)][D - \Psi_{22}(I, S)]S = 0. \tag{22b}
\]

and then comparing (22) and (8) we obtain conditions on \(\Psi_{ij}\) as

\[
c = -(\Psi_{12} + \Psi_{11} + \frac{\partial \Psi_{11}}{\partial I}I), \quad c = -(\Psi_{21} + \Psi_{22} + \frac{\partial \Psi_{22}}{\partial S}S) \tag{23}
\]

and

\[
\Psi_{12}\Psi_{11} = (S - \lambda) \quad \text{and} \quad \Psi_{21}\Psi_{22} = -I.
\]

Now, choosing \(\Psi_{ij}\) such that

\[
\Psi_{12} = \frac{1}{q_1}(S - \lambda), \quad \Psi_{11} = q_1, \quad \Psi_{21} = -\frac{1}{q_2}I, \quad \Psi_{22} = q_2. \tag{24}
\]
where \( q_1, q_2 \) are arbitrary constants must be determined.

The first condition in equation (23) leads to
\[
\frac{1}{q_1} S + (q_1 - \frac{1}{q_1} \lambda) = -c, \quad q_2 - \frac{1}{q_2} I = -c,
\]
then
\[
q_1 = \frac{-c \pm \sqrt{c^2 + 4\lambda}}{2}, \quad q_2 = -c,
\] (25)
and system (22) takes the form
\[
[D - \frac{1}{q_1} (S - \lambda)][D - q_1]I = 0, \quad [D + \frac{1}{q_2} ][D - q_2]S = 0. \quad (1)
\]
compatible first order system is
\[
I' - q_1 I = 0, \quad S' - q_2 S = 0 \quad (26)
\]
Integrating (26) and using (25) gives the particular solution of (18)
\[
I(z) = \exp[q_1(z - z_0)], \quad S(z) = \exp[q_2(z - z_0)] \quad (27)
\]
where \( z_0 \) is the integration constant.

3.3 Predator-Prey two component telegraph model

The predator-Prey two component system is given as
\[
\tau \frac{\partial^2 u}{\partial t^2} + (1 - \tau \frac{\partial f}{\partial u}) \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f_1(u, v), \quad f_1(u, v) = u \left( a_1 - b_1 v \right), \quad (29a)
\]
\[
\tau \frac{\partial^2 v}{\partial t^2} + (1 - \tau \frac{\partial f}{\partial v}) \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f_2(u, v), \quad f_2(u, v) = v \left( a_2 - b_2 u \right), \quad (29b)
\]
where \( \tau \) is the time delay constant. Using the coordinate transformation \( z = x - ct \) in system (29) we obtain the following nonlinear ordinary differential system of two equations
\[
u'' + g_1(u, v)u' + f_1(u, v) = 0, \quad v'' + g_1(u, v)v' + f_2(u, v) = 0. \quad (30)
\]
where
\[
g_1(u, v) = c\beta((1 + \tau a_1) + b_1 v), \quad f_1(u, v) = \beta u \left( a_1 - b_1 v \right), \quad (31a)
\]
\[
g_2(u, v) = c\beta((1 + \tau a_2) + b_2 u), \quad f_2(u, v) = \beta v \left( a_2 - b_2 u \right). \quad (31b)
\]
Factorization of (3) leads to
\[
[D - \Psi_{12}(u, v)][D - \Psi_{11}(u, v)]u = 0, \quad [D - \Psi_{21}(u, v)][D - \Psi_{22}(u, v)]v = 0.
\] (32)

Now, we choose \(\Psi_{ij}\) such that

\[
\Psi_{12} = \frac{1}{k_1}\sqrt{\beta(a_1 - b_1v)}, \quad \Psi_{11} = \sqrt{\beta k_1}, \quad (33a)
\]

\[
\Psi_{21} = \frac{1}{k_2}\sqrt{\beta(a_2 - b_2u)}, \quad \Psi_{22} = \sqrt{\beta k_2}. \quad (33b)
\]

where \(k_1\) and \(k_2\) are arbitrary constants must be determined.

The first condition in equation (8) leads to

\[
k_1 = \frac{-c\sqrt{\beta(1 + \tau a_1)} \pm \sqrt{c^2\beta(1 + \tau a_1)^2 - 4a_1}}{2}, \quad (34a)
\]

\[
k_2 = \frac{-c\sqrt{\beta(1 + \tau a_2)} \pm \sqrt{c^2\beta(1 + \tau a_2)^2 - 4a_2}}{2}. \quad (34b)
\]

The compatible first order system is

\[
u' - \sqrt{\beta k_1}u = 0, \quad v' - \sqrt{\beta k_2}v = 0 \quad (35)
\]

Integrating (35) and use (34) gives the particular solution of (29)’

\[
u(z) = \exp[\sqrt{\beta k_1}(z - z_0)], \quad \theta(z) = \exp[\sqrt{\beta k_2}(z - z_0)]. \quad (36)
\]

4 Conclusions

In conclusion, the efficient factorization method for system of two nonlinear second order ordinary differential equation was introduced and has been applied to find explicit exact particular travelling wave solutions for three deferent models of system of equations.

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References


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