A Control Trajectory Problem: 
Continuous Systems

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Abstract

In this paper we consider a fundamental control problem. Our aim is not to determine the control which steers the system to a desired final state at time T, but to investigate, under some hypothesis, the input which makes the trajectory of the system, along the interval of time \([0, T]\), to be exactly equal to a desired given one. To resolve the problem, we use a state space technique (see [1, 2, 3]), generally used in the analysis and control of hereditary systems. We also study the regional aspect of the problem.

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1 Introduction

Consider the linear system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) , \text{ } t \in [0, T] \\
x(0) &= x_0,
\end{align*}
\]

(1)

where \(A\) is the generator of a strongly continuous semi-group \((R(t))_{t \geq 0}\) on the Hilbert space \(X\), \(B \in \mathcal{L}(U, X)\), where \(U\) is a Hilbert space.
Given a desired trajectory \( x_d(.) \in L^2(0,T;X) \), we investigate \(^1\) the control law \( u \), having a minimal norm, which allows the state \( x(t) \) to coincide with \( x_d(t) \) along the interval of time \([0,T] \), i.e.,

\[
\begin{align*}
\{ x(.,x_0,u) & = x_d(.) \text{ on } L^2(0,T;X) \\
\|u\| & = \min \{ \|v\| : x(.,x_0,v) = x_d \}
\end{align*}
\]

where \( x(.,x_0,v) \) is the state of system 1 corresponding to the command \( v \).

1.1 State space technique

We consider the strongly continuous semi-group \((S(t))_{t \geq 0}\) defined on the Hilbert space \( Y = L^2(-T,0;X) \) by

\[
(S(t)y)(r) = \begin{cases} 
y(t+r) & \text{if } r \in [-T,-t] \\
0 & \text{if } r \in [-t,0]
\end{cases}, \quad \text{pour } t \leq T
\]

and

\[(S(t)y)(r) = 0; \forall r \in [-T,0] , \forall t > T.\]

The generator of \((S(t))_{t \geq 0}\) is the operator \( D = \frac{d}{dt} \) with domaine \( \text{Dom}(D) = \{ y \in W^{1,2}(-T,0;X) : y(0) = 0 \} \). Let \( F \in \mathcal{L}(X,Y) \) be the operator defined by

\[(Fx)(r) = x , \forall x \in X , \forall r \in [-T,0].\]

For every \( x_0 \in X \) and all control \( u \in L^2(0,T;U) \), we define on \( Y \), the following evolution system

\[
y(t,x_0,u) = S(t)(Fx_0) - D \int_0^t S(t-r)Fx(r,x_0,u)dr , \quad t \in [0,T]
\]

where \( x(.,x_0,u) \) is the solution of equation (1) corresponding to the control \( u \).

Remark 1.1 \([4],[3]\]

i) For every \( t \leq T \)

\[(y(t,x_0,u))(r) = \begin{cases} 
x_0 & \text{if } r \in [-T,-t] \\
x(t+r,x_0,u) & \text{if } r \in [-t,0]
\end{cases}, \quad \text{hence, } (y(T,x_0,u))(r) = x(T+r,x_0,u) , \forall r \in [-T,0].\]

ii) For every \( t \leq T \)

\[(-D \int_0^t S(t-r)Fx(r,x_0,u)dr)(s) = \begin{cases} 
x(t+s,x_0,u) & \text{if } s \in [-t,0] \\
0 & \text{if } s \in [-T,-t]
\end{cases} \]

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The introduction of the state \( y(.,x_0,u) \) allows us to transform the control trajectory problem to a standard controllability problem on the space \( Y \). Indeed, to determine a control \( u \) such that \( x(.,x_0,u) = x_d(.) \) is equivalent to find a control \( u \) which satisfy \( y(T) = y_d \), where \( y_d \in Y \) is the desired state defined by \( y_d(.) = x_d(.-T) \).

The input \( u \) acts on the signal \( y(.,x_0,u) \) in a non standard form, this suggest us to define an other evolution equation on the state \( Z = X \times Y \).

Precisely, we consider the system defined on the Hilbert space \( Z = X \times Y \),

\[
 z(t) = U(t)z_0 + \int_0^t U(t-r)Lu(r)dr , \forall t \in [0,T]. \tag{3}
\]

where \( L = (B,0)^T \), \( z_0 = (x_0,Fx_0)^T \) and \( (U(t))_{t \geq 0} \) the strongly continuous semi-group defined by

\[
 U(t) = \begin{pmatrix}
 U_0(t) & 0 \\
 U_1(t) & U_2(t)
\end{pmatrix} ; \forall t \geq 0 \tag{4}
\]

where

\[
 U_0(t) = R(t), U_2(t) = S(t) \text{ and } U_1(t) = -D \int_0^t S(t-r)FR(r)dr.
\]

**Remark 1.2** We verify easily that for all \( x_0 \in X \) and all control \( u \in L^2(0,T;U) \), we have

\[
 z(.,x_0,u) = (x(.,x_0,u),y(.,x_0,u)), \tag{5}
\]

where \( z(.,x_0,u) \) is the solution of (3).

**1.2 Fundamental results**

Define the operator \( H \) by

\[
 H : \begin{array}{c}
     L^2(0,T;U) \\
     u
\end{array} \rightarrow \begin{array}{c}
     Y \\
     \rightarrow p_2 \int_0^T U(T-r)Lu(r)dr
\end{array}
\]

where \( p_2 \) is the operator

\[
 p_2 : \begin{array}{c}
     Z \\
     (x,y)
\end{array} \rightarrow Y
\]

and \( (U(t))_{t \geq 0} \) is the strongly continuous semi group defined by (4).

**Remark 1.3** the operator \( H \) is bounded and have an adjoint operator given by

\[
 H^* : \begin{array}{c}
     Y \\
     y
\end{array} \rightarrow \begin{array}{c}
     L^2(0,T;U) \\
     \rightarrow B^* \int_{-T}^T R^*(T-.+r)y(r)dr.
\end{array}
\]
Let’s consider the Hilbert space

\[ F_0 = \overline{\text{Im} H} = (\text{Ker} \ H^*)^\perp \]  

(6)

and the semi norm defined on \( Y \) by

\[ \|f\|_F = \|H^*f\|_{L^2(0, T; U)} \]

The corresponding scalar product to this semi norm is

\[ \langle f, g \rangle = \langle H^*f, H^*g \rangle \quad \forall f, g \in Y. \]

**Remark 1.4** We easily establish that

i) \( \|\cdot\|_F \) is a norm on \( F_0 \).

ii) \((HH^*)(Y) \subset F_0\).

Define the operator \( \Lambda \) by

\[ \Lambda : F_0 \to F_0 \]

\[ f \to HH^*f. \]

It follows from remark 1.4 that \( \Lambda \) is well defined and bounded. If \( F \) is the completion space of \( F_0 \) respectively to the norm \( \|\cdot\|_F \), then operator \( \Lambda \) has a unique extension, denoted also \( \Lambda \), which is an isomorphism from the space \( F \) to its dual \( F' \).

Let’s define the operator \( G \) by

\[ G : L^2(0, T; X) \to L^2(-T, 0; X) \]

\[ y \to y(T + .) \]

**Proposition 1.1** Let \( x_0 \in X \) and \( y_d \in L^2(0, T; X) \) a given desired trajectory. If \( y_d \in G^{-1}(p_2(U(T)z_0) + F') \), where \( z_0 = (x_0, Fx_0)^T \), then there exists a unique control \( u^* \in L^2(0, T; U) \) such that

\[ \begin{cases} x(., x_0, u^*) = y_d(.) \text{ on } L^2(0, T; X) \\ \|u^*\| = \min \{ \|v\| : x(., x_0, v) = y_d \}. \end{cases} \]

The input \( u^* \) is given by

\[ u^* = H^*f, \]  

(7)

where \( f \) is the unique solution of the equation

\[ \Lambda f = G y_d - p_2(U(T)z_0). \]

(8)

Proof. Since \( y_d \in G^{-1}(p_2(U(T)z_0) + F') \), then

\[ y_d(T + .) \in p_2(U(T)z_0) + F'. \]
but Λ is an isomorphism, hence there exists a unique \( f \in F \) such that
\[
y_d(T + \cdot) = p_2(U(T)z_0) + \Lambda f.
\]
Consider the control \( u^* = H^* f \), we can write \( y_d(T + \cdot) \) as follows
\[
y_d(T + \cdot) = p_2(U(T)z_0) + H u^*
\]
which implies that \( y_d(T + \cdot) = p_2(z(T, x_0, u^*)) \), and from remark 1.2 we deduce that \( y_d(T + \cdot) = y(T, x_0, u^*) \). Finally, from remark 1.1 we obtain
\[
y_d = x(., x_0, u^*).
\]
On the other hand, to proof the optimality of \( u^* \), we consider the set
\[
C = \{ v \in L^2(0, T; U) : x(., x_0, v) = y_d \}.
\]
For every \( v \in C \), we have \( x(., x_0, v) = x(., x_0, u^*) \). Hence \( Hu^* = Hv \), consequently
\[
< H(v - u^*), f > = 0, \text{ i.e., } < v - u^*, u^* > = 0, \text{ thus } ||u^*|| \leq ||v||.
\]
To complete the precedent result, we give a characterization of the reachable trajectory.

**Proposition 1.2** Let \( W = \{ x(., x_0, u) : u \in L^2(0, T; U) \} \) be the set of all reachable trajectories, from an initial state \( x_0 \), on \([0, T] \), then
\[
W = G^{-1}(p_2(U(T)z_0) + F').
\]
Proof. It follows from proposition 1.1 that
\[
W \supset G^{-1}(p_2(U(T)z_0) + F').
\]
Inversely, given \( x(., x_0, u) \in W \), we consider the linear form
\[
\Psi : F_0 \to \mathbb{R}
\]
\[
f \mapsto < Gx(., x_0, u) - p_2(U(T)z_0), f > .
\]
From the density of \( F_0 \) on \( F \), we deduce that
\[
|\Psi(f)| \leq ||u|| \cdot ||f||, \forall f \in F.
\]
Finally, it follows from the Riesz theorem that \( Gx(., x_0, u) - p_2(U(T)z_0) \in F' \).

**Remark 1.5** Similarly to what was done by Emirsajlow in [5] and [6], the approach developped in this section can be used to resolve the following control problem
\[
\min \{ \int_0^T < u(t), Ru(t) > dt : u \in V_\alpha \},
\]
where
\[
V_\alpha = \{ u \in L^2(0, T; U) : ||x(., x_0, u) - y_d||_{L^2(0, T; X)} \leq \alpha \}, (\alpha > 0)
\]
and \( R \) a self-adjoint definite positif operator.
1.3 A finite dimensional case

Consider the system
\[
\begin{cases}
\dot{x}(t) = Ax(t) + Bu(t), & t \in [0, T] \\
x(0) = x_0
\end{cases}
\] (9)

where $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^n)$.

**Proposition 1.3** If there exists a matrix $K_2 \in \mathcal{L}(\mathbb{R}^n)$ such that the matrix $BK_2$ is invertible, then the space $F_0$ defined by (6) is given by
\[
F_0 = L^2(-T, 0; \mathbb{R}^n).
\]

**Proof.** We take $y_d \in L^2(-T, 0; \mathbb{R}^n)$, $\epsilon > 0$ and seek a control $u$ such that
\[
||Hu - y_d|| \leq \epsilon.
\]

Let $K_1$ be an arbitrary matrix in $\mathcal{L}(\mathbb{R}^n)$ and $h \in ]0, T]$. We easily deduce from [7] that, under the invertibility hypothesis of the matrix $BK_2$, the system
\[
\begin{cases}
\dot{q}(t) = Aq(t) + BK_1q(t - h) + BK_2v(t), & t \in [0, T] \\
q(0) = q_0 \\
q(r) = \Phi_1(r), & \forall r \in [-T, 0]
\end{cases}
\] (10)
is approximately controllable on the space $M^2 = \mathbb{R}^n \times L^2(-T, 0; \mathbb{R}^n)$. In other words, for every $x_d = (a_d, b_d) \in M^2$, there exists a control $v$ in $L^2(0, T; \mathbb{R}^n)$ such that
\[
||q(T, \Phi, v) - x_d||_{M^2} \leq \epsilon,
\]
where $\Phi = (q_0, \Phi_1) \in M^2$ is the initial state, $q(., \Phi, v)$ is the state variable corresponding to $\Phi$ and the input $v$, $q_T(., \Phi, v) \in L^2(-T, 0; \mathbb{R}^n)$ is defined by
\[
[q_T(., \Phi, v)](r) = q(T + r, \Phi, v), \ r \in [-T, 0].
\]

It follows from equation (11) that given an initial state $\Phi = (x_0, \Phi_1)$, where $x_0$ is the initial state of system 9 and $\Phi_1 \in L^2(-T, 0; \mathbb{R}^n)$, given a desired state $c_d = (x_d, y_d + p_2(U(T)z_0))$, where $x_d$ is an arbitrary element of $\mathbb{R}^n$ and $z_0 = (x_0, Fx_0)$, there exists a control $v^* \in L^2(0, T; \mathbb{R}^n)$ such that
\[
||q(T, \Phi, v^*), q_T(., \Phi, v^*)) - c_d||_{M^2} \leq \epsilon,
\]

hence
\[
||q_T(., \Phi, v^*) - y_d - p_2(U(T)z_0)||_{L^2(-T, 0; \mathbb{R})} \leq \epsilon.
\]

Then define the control variable
\[
u_{v^*}(t) = K_1 q(t - h, \Phi, v^*) + K_2 v^*(t).
\]
Since \( q(\cdot, \Phi, v^*) \) is the solution of equation
\[
\dot{q}(t) = Aq(t) + BK_1q(t-h) + BK_2v^*(t), \quad t \in [0, T],
\]
we deduce that
\[
\begin{align*}
\dot{q}(t) &= Aq(t, \Phi, v^*) + Bu_v(t), \quad t \in [0, T] \\
q(0, \Phi, v^*) &= x_0,
\end{align*}
\]
thus
\[
q(t, \Phi, v^*) = x(t, x_0, u_v^*), \quad r \in [0, T]
\]
consequently,
\[
q(T + r, \Phi, v^*) = x(T + r, x_0, u_v^*), \quad t \in [-T, 0].
\]
From remark 1.1, we have \( q_T(\cdot, \Phi, v^*) = y(T, x_0, u_v^*) \), hence
\[
\|y(T, x_0, u_v^*) - y_d - p_2(U(T)z_0)\|_{L^2(-T, 0; IR)} \leq \epsilon.
\]
Finally, from remark 1.2 we obtain that
\[
y(T, x_0, u_v^*) = p_2(z(T, x_0, u_v^*)), \text{ and then } \|Hu_v - y_d\| \leq \epsilon.
\]

2 A regional control trajectory problem

In this section we study the regional aspect of the control trajectory problem, i.e., we suppose that the state space is \( X = L^2(\Omega) \), where \( \Omega \) is an open bounded subset of \( IR^n \), and we consider a region \( \omega \subset \Omega \) and a desired trajectory \( y_d \in L^2(0, T; L^2(\omega)) \), then we investigate the control \( u \) solution of the problem
\[
\begin{align*}
[x(\cdot, x_0, u)] &= y_d \\
\|u\| &= \min\{\|v\| : [x(\cdot, x_0, v)]/\omega = y_d\}.
\end{align*}
\]

2.1 Definition of the problem-caracterization

Given a region \( \omega \) of \( \Omega \), we consider the bounded operators \( M_\omega, M \) and \( H_\omega \) defined by
\[
\begin{align*}
M_\omega : & \text{ } L^2(-T, 0; X) \rightarrow L^2(-T, 0; L^2(\omega)) \\
& f \rightarrow f/\omega \\
M : & \text{ } Z = X \times Y \rightarrow L^2(-T, 0; L^2(\omega)) \\
& (f, g) \rightarrow M_\omega(g) \\
H_\omega : & \text{ } L^2(0, T; U) \rightarrow L^2(-T, 0; L^2(\omega)) \\
& u \rightarrow M(\int_0^T U(T-r)Lu(r)dr).
\end{align*}
\]
and the Hilbert space $E_0 = \overline{\text{im} \, H_\omega} = (\text{Ker} \, H_\omega^*)^\perp$.

We define on $L^2(-T, 0; L^2(\omega))$, the semi norm
$$||f||_{E_0} = ||H_\omega^* f||_{L^2(0, T; U)}$$
and the corresponding inner product by
$$\langle \langle f, g \rangle \rangle_{E_0} = \langle H_\omega^* f, H_\omega^* g \rangle, \quad \forall f, g \in L^2(-T, 0; L^2(\omega))$$

**Remark 2.1** We have

(i) $||.||_{E_0}$ is a norm on the space $E_0$.

(ii) $(H_\omega H_\omega^*)(L^2(-T, 0; L^2(\omega))) \subset E_0$.

Define the operator $\Lambda_\omega$ by
$$\Lambda_\omega : E_0 \to E_0 \quad f \mapsto H_\omega H_\omega^* f.$$

It follows from the precedent remark that $\Lambda_\omega$ is well defined, we also verify easily that it is bounded.

Let $E$ be the completion space of $E_0$ relatively to the norm $||.||_{E_0}$. The operator $\Lambda_\omega$ can be extended continuously, and uniquely, to an isomorphism defined from $E$ to its dual $E'$. This extension is also denoted $\Lambda_\omega$.

To establish the fundamental result of this section we introduce the operator $G_\omega$ defined by
$$G_\omega : L^2(0, T; L^2(\omega)) \to L^2(-T, 0; L^2(\omega)) \quad y \mapsto y(T + .)$$
$G_\omega$ is bijectif and has an inverse operator described by
$$G_\omega^{-1} : L^2(-T, 0; L^2(\omega)) \to L^2(0, T; L^2(\omega)) \quad y \mapsto y(-T)$$

**Proposition 2.1** Let $x_0 \in X$ and $y_d \in L^2(0, T; L^2(\omega))$ a desired given trajectory, if $y_d \in G_\omega^{-1}(M(U(T)z_0) + E')$, then there exists a unique control $u^* \in L^2(0, T; U)$ such that
$$\begin{cases}
[x(., x_0, u^*)]/\omega = y_d(.) \text{ in } L^2(0, T; L^2(\omega)) \\
||u^*|| = \min\{||v|| : [x(., x_0, v)]/\omega = y_d\}.
\end{cases}$$

$u^*$ is given by
$$u^* = H_\omega^* f \quad (14)$$
where $f$ is the unique solution of equation
$$\Lambda_\omega f = G_\omega y_d - M(U(T)z_0). \quad (15)$$

Moreover, the set $W_\omega = \{(x(., x_0, u))/\omega : u \in L^2(0, T; U)\}$ of all trajectories $\omega$-reachable on $[0, T]$ is given by
$$W_\omega = G_\omega^{-1}(M(U(T)z_0) + E').$$

**Proof.** The proof is similar to the ones of proposition 1.1 and 1.2. 

\[\blacksquare\]
2.2 Application

Let $\omega$ and $\bar{\omega}$ be given regions of $\Omega$, $y_d \in L^2(\bar{\omega})$ a desired state and $z_d(.) \in L^2(0, T; L^2(\omega))$ a desired trajectory. The regional control trajectory problem consists of determining, under some hypothesis, the control $u^*$ solution of the following problem

$$||u^*|| = \min ||v||$$

where $v$ verify

$$\begin{cases} x(T, x_0, v)/\bar{\omega} = y_d \\ x(., x_0, v)/\omega = z_d(.) \text{ dans } L^2(0, T; X) \end{cases}$$

To resolve this problem, we define the following operators $P : X \times Y \to L^2(\bar{\omega})$ and $H_{\bar{\omega}, \omega} : L^2(0, T; U) \to L^2(\bar{\omega}) \times L^2(-T, 0; L^2(\omega))$, such that

$$H_{\bar{\omega}, \omega}(u) = (P(\int_0^T U(T-r)Lu(r)dr), M(\int_0^T U(T-r)Lu(r)dr))$$

where the operator $M$ is defined by equation (13).

Consider the Hilbert space $N_0 = \text{Im} H_{\bar{\omega}, \omega} = (\text{Ker} H_{\bar{\omega}, \omega}^*)^\perp$ and define on the space $L^2(\omega) \times L^2(-T, 0; L^2(\omega))$ the semi norm

$$||f||_N = ||H_{\bar{\omega}, \omega}^* f||_{L^2(0, T; U)}.$$

**Remark 2.2**

i) $||.||_N$ is a norm on the space $N_0$.

ii) $(H_{\bar{\omega}, \omega}^* H_{\bar{\omega}, \omega})(L^2(\omega) \times L^2(-T, 0; L^2(\omega))) \subset N_0$.

We deduce from the above that the operator $\Lambda_{\bar{\omega}, \omega}$ defined by

$$\Lambda_{\bar{\omega}, \omega} : N_0 \to N_0$$

$$f \to (H_{\bar{\omega}, \omega}^* H_{\bar{\omega}, \omega})(f)$$

is bounded and well defined.

Let $N$ be the completion space of $N_0$ respectively to the norm $||.||_N$. The operator $\Lambda_{\bar{\omega}, \omega}$ can be extended continuously and uniquely to an isomorphism defined from $N$ to its dual space $N'$. This extension is also denoted by $\Lambda_{\bar{\omega}, \omega}$.

Define the operator $K_{\bar{\omega}, \omega}$ by

$$K_{\bar{\omega}, \omega} : L^2(\bar{\omega}) \times L^2(-T, 0; L^2(\omega)) \to L^2(\bar{\omega}) \times L^2(-T, 0; L^2(\omega))$$

$$y, z(.) \to (y, z(.-T)).$$

$K_{\bar{\omega}, \omega}$ is bijectif and its inverse operator is given by

$$K_{\bar{\omega}, \omega}^{-1}(y, z(.)) = (y, z(.-T)).$$
Proposition 2.2

1) Let $x_0 \in X, y_d \in L^2(\bar{\omega})$ and $z_d \in L^2(0, T; L^2(\omega))$. If $(y_d, z_d) \in K_{\bar{\omega}}^{-1}((P(U(T)z_0), M(U(T)z_0)) + N')$, then there exists a unique control $u^* \in L^2(0, T; U)$ solution of the problem, and $u^*$ is given by

$$u^* = H^*_{\bar{\omega}} f,$$

where $f$ is the unique solution of the equation

$$A_{\bar{\omega}} f = K_{\bar{\omega}} (y_d, z_d) - (P(U(T)z_0), M(U(T)z_0)).$$

2) The set

$$Q = \{([x(., x_0, u)]/\bar{\omega}, [x(., x_0, u)]/\omega) : u \in L^2(0, T; U)\}$$

is equal to the set

$$K_{\bar{\omega}}^{-1}((P(U(T)z_0), M(U(T)z_0)) + N').$$

Proof. The proof is similar to the ones of propositions 1.1, 1.2.

References


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