Weak and Strong Convergence Theorems for $k$-Strictly Pseudo-Contractive in Hilbert Space

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Abstract. Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$, and assume that $T_i : K \rightarrow H, i = 1, 2, \ldots, N$ be a finite family of $k_i$-strictly pseudo-contractive mappings for some $0 \leq k_i \leq 1$ such that $\bigcap_{i=1}^{N} F(T_i) = \{ x \in K : x = T_ix, i = 1, 2, \ldots, N \} \neq \emptyset$. For the following iterative algorithm in $K$, for $x_1, x'_1 \in K$ and $u \in K$,

\[
\begin{align*}
\{y_n &= P_K[kx_n + (1 - k)\sum_{i=1}^{N} \lambda_i T_ix_n] \\
x_{n+1} &= \beta_n x_n + (1 - \beta_n)y_n
\end{align*}
\]

and

\[
\begin{align*}
\{y'_n &= P_K[\alpha'_n x'_n + (1 - \alpha'_n)\sum_{i=1}^{N} \lambda_i T_ix'_n] \\
x'_{n+1} &= \beta'_n u + (1 - \beta'_n)y'_n
\end{align*}
\]

$P_K$ is the metric projection of $H$ onto $K$, $\{\alpha'_n\}$ and $\{\beta'_n\}$ are sequences in $(0,1)$ satisfying appropriate conditions, we proved that $\{x_n\}$ and $\{x'_n\}$ respectively converges strongly to a common fixed point of $\{T_i\}_{i=1}^{N}$. Our results improve and extend the results announced by Genaro L.A. and H.K.Xu [Iterative methods for strict pseudo-contractions in Hilbert spaces, Nonl.Anal.67(2007) 2258-2271], T.H.Kim and H.K.Xu [Strong convergence of modified Mann iterations, Nonlinear Anal.61(2005)51-60] and G.Marino and H.K.Xu [Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J.Math.Anal.Appl.329(2007)336-346].
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1. Introduction

Let \(K\) be a nonempty closed convex subset of a Hilbert space \(H\). We use \(F(T)\) to denote the fixed point set of \(T\) and \(P_K\) to denote the metric projection of \(H\) onto \(K\). Recall that a mapping \(T: K \to H\) is said to be a \(k\)-strictly pseudo-contractive if there exists a constant \(k \in [0, 1)\) such that

\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \forall x, y \in K
\]  

(1.1)

Note that the class of \(k\)-strictly pseudo-contractions includes strictly the class of nonexpansive mappings which are mappings \(T\) on \(K\) such that

\[
\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in K.
\]

When \(k = 0\), \(T\) is said to be nonexpansive, and it is said to be pseudo-contractive if \(k = 1\). \(T\) is said to be strongly pseudo-contractive if there exist a positive constant \(\lambda \in (0, 1)\) such that \(T - \lambda I\) is pseudo-contractive. Clearly, the class of \(k\) strict pseudo-contraction falls into the one between classes of nonexpansive mappings and pseudo-contractions. We remark also that the class of strongly pseudo-contractive mappings is independent of the class of \(k\) strict pseudo-contraction (see [2, 3, 5]).

It is very clear that, in a real Hilbert space \(H\), (1.1) is equivalent to

\[
\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - k}{2}\|(x - Tx) - (y - Ty)\|^2, \forall x, y \in K.
\]

(1.2)

\(T\) is pseudo-contractive if and only if

\[
\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2
\]

(1.3)

\(T\) is strongly pseudo-contractive if and only if there exists a positive constant \(\lambda \in (0, 1)\) such that

\[
\langle Tx - Ty, x - y \rangle \leq (1 - \lambda)\|x - y\|^2, \forall x, y \in K.
\]

(1.4)

Recall that the normal Mann’s iterative algorithm was introduced by Mann (see [1]) in 1953. Since then, construction of fixed points for nonexpansive mapping have been extensively investigated (see [4, 8, 9, 12, 14, 17, 18, 19, 20, 21]) and \(k\) strict pseudo-contractions via the normal Mann’s iterative algorithm has been extensively investigated by many authors (see [1, 7, 13, 15, 16, 22, 23]).

The normal Mann’s iterative algorithm generates a sequence \(\{x_n\}\) in the following manner:

\[
\forall x_1 \in K, x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, n \geq 1
\]

(1.5)
In 1967, Browder and Petryshyn [5] established the first convergence result for $k$-pseudo-contractive self mappings in real Hilbert spaces. They prove weak and strong convergence theorems by using algorithm (1.5) with a constant control sequence $\{\alpha_n\} \equiv \alpha$ for all $n$. Afterward, Rhoades [6] generalized in part the corresponding results in [5] in the sense that a variable control sequence $\{\alpha_n\}$ was taken into consideration. Under the assumption that the domain of mapping $T$ is compact convex, he established a strong convergence theorem by using algorithm (1.5) with a control sequence $\{\alpha_n\}$ satisfying the conditions $\alpha_1 = 1, 0 < \alpha_n < 1, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n \to \infty} \alpha_n = \alpha < 1 - k$. However, without the compact assumption on the domain of mapping $T$, in general, one cannot expect to infer any weak convergence results from Rhoades’ convergence theorem.

Very recently, G.L. Acedo and Xu [24] have proved a weak convergence theorem by using algorithm (1.6) with certain control conditions.

In this paper, motivated by G.L. Acedo and Xu [24] and the above results, we study the following iteration process (1.7) and (1.8), for $x_1 \in K$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \sum_{i=1}^{N} \lambda_i T_i x_n$$

with certain control conditions.

In this paper, motivated by G.L. Acedo and Xu [24] and the above results, we study the following iteration process (1.7) and (1.8), for $x_1 \in K$,

$$\begin{cases} y_n = P_K[kx_n + (1 - k) \sum_{i=1}^{N} \lambda_i T_i x_n] \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n \end{cases}$$

and

$$\begin{cases} y'_n = P_K[\alpha'_n x'_n + (1 - \alpha'_n) \sum_{i=1}^{N} \lambda_i T_i x'_n] \\ x'_{n+1} = \beta'_n u_n + (1 - \beta'_n) y'_n \end{cases}$$

$P_K$ is the metric projection of $H$ onto $K$, $\{\alpha'_n\}$ and $\{\beta'_n\}$ are sequences in (0,1) satisfying appropriate conditions, we proved that $\{x_n\}$ and $\{x'_n\}$ respectively converges strongly to a common fixed point of $\{T_i\}_{i=1}^{N}$. Our results extend and improve the corresponding results in [19, 23, 24].

We will use the following notation:
1. $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence.
2. $\omega_{\omega}(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak $\omega$-limit set of $\{x_n\}$.

2. Preliminaries

We need some Lemmas and Propositions in real Hilbert space $H$, which are listed as follow:

**Lemma 2.1.** (Marino and Xu [23]) Let $H$ be a real Hilbert space, there hold the following identities.

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y \rangle + \|y\|^2, \forall x, y \in H$

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \forall t \in [0, 1], \forall x, y \in H$
Lemma 2.2. (Demiclosedness Principle). If $T$ is $k$-strict pseudo-contraction on closed convex subset $K$ of a real Hilbert space $H$, then $I - T$ is demiclosed at any point $y \in H$.

Lemma 2.3. (Xu [23]). Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there satisfies $\langle x - y, y - z \rangle \geq 0 \forall z \in C$.

Lemma 2.4. (see, e.g. Liu [11]). Let $\{a_n\}$ be a sequence of nonnegative real numbers that satisfies the condition

$$a_{n+1} \leq (1 - t_n)a_n + b_n + 0(t_n), n \geq 1,$$

where $\{t_n\}$ satisfies the restrictions:

(i) $t_n \to 0 (n \to \infty)$;

(ii) $\sum_{n=1}^{\infty} b_n < \infty$;

(iii) $\sum_{n=1}^{\infty} t_n = \infty$.

then $a_n \to 0$ as $n \to \infty$.

Proposition 2.5. Assume $K$ is closed convex subset of Hilbert space $H$.

(i) Given an integer $N \geq 1$, assume, for each $1 \leq i \leq N$, $T_i : K \to H$ is a $k_i$-strict pseudo-contraction for some $0 \leq k_i < 1$. Assume $\{\lambda_i\}_{i=1}^{N}$ is a positive sequence such that $\sum_{i=1}^{N} \lambda_i = 1$. Then $\sum_{i=1}^{N} \lambda_i T_i$ is a $k$-strict pseudo-contraction, with $k = \max \{k_1 : 1 \leq i \leq N\}$.

(ii) Let $\{T_i\}_{i=1}^{N}$ and $\{\lambda_i\}_{i=1}^{N}$ be given as in (i) above. Suppose that $\{T_i\}_{i=1}^{N}$ has a common fixed point. Then

$$Fix(\sum_{i=1}^{N} \lambda_i T_i) = \bigcap_{i=1}^{N} Fix(T_i).$$

Proof. To prove (i), we only need to consider the case of $N = 2$. the general case can be proved by induction. Set $A = (1 - \lambda)T_1 + \lambda T_2$, where $\lambda \in (0, 1)$ and for $i = 1, 2$, $T_i$ is a $k_i$-strict pseudo-contraction. Set $k = \max \{k_1, k_2\}$. We now to prove that $A$ is a $k$-strict pseudo-contraction, by lemma 2.1(ii) we have

$$\|(I - A)x - (I - A)y\|^2$$

$$= \|(1 - \lambda)((I - T_1)x - (I - T_1)y) + \lambda((I - T_2)x - (I - T_2)y)\|^2$$

$$= (1 - \lambda)\|((I - T_1)x - (I - T_1)y\|^2 + \lambda\|((I - T_2)x - (I - T_2)y\|^2$$

$$\lambda(1 - \lambda)\|((I - T_1)x - (I - T_1)y - [(I - T_2)x - (I - T_2)y]\|^2$$

and observe that $T : K \to H$ is a $k$-strict pseudo-contraction if and only if there holds the following

$$\langle x - y, (I - T)x - (I - T)y \rangle \geq \frac{1 - k}{2}\|(I - T)x - (I - T)y\|^2$$

Indeed, putting $V = I - T$, we see that (1.1) holds if and only if

$$\|(I - V)x - (I - V)y\|^2 \leq \|x - y\|^2 + k\|Vx - Vy\|^2, \forall x, y \in K$$

(2.3)
But by lemma 2.1(i) we have
\[ \| (I - V)x - (I - V)y \|^2 = \| x - y \|^2 - 2\langle x - y, Vx -Vy \rangle + \| Vx - Vy \|^2 \] (2.4)
substituting (2.4) into (2.3), we obtain (2.2). Noticing (2.1), we have
\[ \langle x - y, (I - A)x - (I - A)y \rangle = (1 - \lambda)\langle x - y, (I - T_1)x - (I - T_1)y \rangle + \lambda\langle x - y, (I - T_2)x - (I - T_2)y \rangle \]
\[ \geq \frac{1 - k}{2} [(1 - \lambda)\| (I - T_1)x - (I - T_1)y \|^2 + \lambda\| (I - T_2)x - (I - T_2)y \|^2] \]
\[ \geq \frac{1 - k}{2} \| (I - A)x - (I - A)y \|^2 \]
Hence A is a k-strict pseudo-contraction.

To prove (ii), we can assume N = 2. It suffices to prove that Fix(A) ⊂ Fix(T_1) ∩ Fix(T_2), where A = (1 - \lambda)T_1 + \lambda T_2, with 0 < \lambda < 1. Let x ∈ Fix(A) and write A_1 = I - T_1 and A_2 = I - T_2.

Take z ∈ Fix(T_1) ∩ Fix(T_2) to deduce that
\[ \| z - x \|^2 = \| (1 - \lambda)(z - T_1x) + \lambda(z - T_2x) \|^2 \]
\[ = (1 - \lambda)\| z - T_1x \|^2 + \lambda\| z - T_2x \|^2 - \lambda(1 - \lambda)\| T_1x - T_2x \|^2 \]
\[ \leq (1 - \lambda)(\| z - x \|^2 + k\| x - T_1x \|^2) \]
\[ + \lambda(\| z - x \|^2 + k\| x - T_2x \|^2) - \lambda(1 - \lambda)\| T_1x - T_2x \|^2 \]
\[ = \| z - x \|^2 + k[(1 - \lambda)\| A_1x \|^2 + \lambda\| A_2x \|^2] - \lambda(1 - \lambda)\| A_1x - A_2x \|^2. \]
It follows that
\[ \lambda(1 - \lambda)\| A_1x - A_2x \|^2 \leq k[(1 - \lambda)\| A_1x \|^2 + \lambda\| A_2x \|^2] \] (2.5)
Since (1 - \lambda)A_1x + \lambda A_2x = 0, we have
\[ (1 - \lambda)\| A_1x \|^2 + \lambda\| A_2x \|^2 = \lambda(1 - \lambda)\| A_1x - A_2x \|^2 \]
This together with (2.5) implies that
\[ (1 - k\lambda)(1 - \lambda)\| A_1x - A_2x \|^2 \leq 0 \]
Since 0 < \lambda < 1 and k < 1, we get \| A_1x - A_2x \| = 0 which implies T_1x = T_2x which in turns implies that T_1x = T_2x = x. Thus, x ∈ Fix(T_1) ∩ Fix(T_2). The general case can be proved by induction, this completes the proof.

**Proposition 2.6.** If T : K → H is a k-strict pseudo-contraction, then T is L-Lipschitzian mapping.

**Proof.** By (1.2), for all x, y ∈ K, we have that
\[ \frac{1 - k}{2} \| (I - T)x - (I - T)y \|^2 \leq \| (I - T)x - (I - T)y, x - y \| \]
\[ \leq \| (I - T)x - (I - T)y \| \| x - y \| \]
it follows that
\[ \| Tx - Ty \| - \| x - y \| \leq \| (I - T)x - (I - T)y \| \]
\[ \leq \frac{2}{1 - k} \| x - y \| , \]
\[ \|Tx - Ty\| \leq L\|x - y\|, \quad L = \frac{3 - k}{1 - k}. \]

**Proposition 2.7.** If \( T \) is a \( k \)-strict pseudo-contraction on a closed convex subset \( K \) of a real Hilbert space \( H \), then the fixed point set \( F(T) \) of \( T \) is closed convex so that the projection \( P_{F(T)} \) is well defined.

**Proof.** Since \( T : K \to H \) is Lipschitzian, we see that \( F(T) \) is closed. Thus, we only need to see that \( F(T) \) is convex; take \( p, q \in F(T) \), and \( t \in (0, 1) \). Put \( z = (1 - t)p + tq \). by using (1.2) we have

\[ \langle z_t - Tz_t, z_t - p \rangle \geq \frac{1 - k}{2}\|z_t - Tz_t\|^2 \]  \hspace{1cm} (2.6)

and

\[ \langle z_t - Tz_t, z_t - q \rangle \geq \frac{1 - k}{2}\|z_t - Tz_t\|^2 \]  \hspace{1cm} (2.7)

Noting that \( z_t - p = t(q - p) \) and \( z_t - q = (1 - t)(p - q) \), substituting these equalities into (2.6) and (2.7), respectively, we get

\[ t\langle z_t - Tz_t, q - p \rangle \geq \frac{1 - k}{2}\|z_t - Tz_t\|^2 \]  \hspace{1cm} (2.8)

and

\[ (1 - t)\langle z_t - Tz_t, p - q \rangle \geq \frac{1 - k}{2}\|z_t - Tz_t\|^2 \]  \hspace{1cm} (2.9)

Multiplied by \( (1 - t) \) and \( t \), and added up on the both sides of (2.8) and (2.9), respectively, we have

\[ \frac{1 - k}{2}\|z_t - Tz_t\|^2 \leq 0, \]

which implies that \( z_t \in F(T) \). This completes the proof.

**Proposition 2.8.** Let \( T : K \to H \) be a \( k \)-strict pseudo-contraction with \( F(T) \neq \emptyset \). Then, \( F(P_KT) = F(T) \).

**Proof.** Clearly, \( F(T) \subset F(P_KT) \). Thus, we only need to show the converse inclusion. Assume that \( x = P_KTx \); then, by lemma 2.1 and lemma 2.3, we have for \( p \in F(T) \) that

\[ \|Tx - p\|^2 = \|Tx - x + x - p\|^2 \]

\[ = \|Tx - x\|^2 + 2\langle Tx - x, x - p \rangle + \|x - p\|^2 \]

\[ = \|Tx - x\|^2 + 2\langle Tx - P_KTx, P_KTx - p \rangle + \|x - p\|^2 \]

\[ \geq \|Tx - x\|^2 + \|x - p\|^2. \]  \hspace{1cm} (2.10)

On the other hand, by (1.1), we have

\[ \|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2. \]  \hspace{1cm} (2.11)

Combining (2.10) and (2.11) yields

\[ (1 - k)\|x - Tx\|^2 \leq 0 \]
Therefore, $x \in F(T)$. This completes the proof.

**Proposition 2.9.** Let $T : K \to H$ be $k$-strict pseudo-contraction. Define $S : K \to H$ by $Sx = \alpha x + (1 - \alpha)Tx$ for each $x \in K$. Then, as $\alpha \in [k, 1)$, $S$ is nonexpansive such that $F(S) = F(T)$.

**Proof.** For all $x, y \in K$, by lemma 2.1(ii) and (1.1) we have

$$
\|Sx - Sy\|^2 = \|\alpha(x - y) + (1 - \alpha)(Tx - Ty)\|^2 \\
= \alpha\|x - y\|^2 + (1 - \alpha)\|Tx - Ty\|^2 \\
- \alpha(1 - \alpha)\|(x - y) - (Tx - Ty)\|^2 \\
\leq \alpha\|x - y\|^2 + (1 - \alpha)\|x - y\|^2 + k(1 - \alpha)\|(x - y) - (Tx - Ty)\|^2 \\
- \alpha(1 - \alpha)\|(x - y) - (Tx - Ty)\|^2 \\
= \|x - y\|^2 - (\alpha - k)(1 - \alpha)\|(x - y) - (Tx - Ty)\|^2 \\
\leq \|x - y\|^2
$$

which proves that $S : K \to H$ is nonexpansive. By the definition of $S$, we have $x - Sx = (1 - \alpha)(x - Tx)$, and this means that $p = Sp$ if and only if $p = Tp$. This completes the proof.

### 3. Main Results

**Theorem 3.1.** Let $K$ be a nonempty closed convex subset of a Hilbert space $H$ and $T_i : K \to H$ be a $k_i$-strictly pseudo-contractive non-self mapping, for some $0 \leq k_i < 1, k = \max\{k_i : 1 \leq i \leq N\}$. Assume the common fixed point set $\bigcap_{i=1}^{N} Fix(T_i)$ is nonempty. Let $\{x_n\}$ be generated by (1.7), i.e.,

$$
x_{n+1} = \beta_n x_n + (1 - \beta_n)P_K[kx_n + (1 - k)\Sigma_{i=1}^{N} \lambda_i T_ix_n]
$$

where $\beta_n = \alpha_n - \frac{k}{1 - k}$, $\{\lambda_i\}_{i=1}^{N}$ is a finite sequence of positive numbers, such that $\Sigma_{i=1}^{N} \lambda_i = 1$ for all $1 \leq i \leq N$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [k, 1]$ and $\Sigma_{i=1}^{N} (\alpha_n - k)(1 - \alpha_n) = \infty$, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^{N}$.

**Proof.** Let $T$ be defined by $T = \Sigma_{i=1}^{N} \lambda_i T_i$, by proposition 2.5 (i),(ii) we know that $Fix(T) = \bigcap_{i=1}^{N} Fix(T_i)$ and $T$ is a $k$-strict pseudo-contraction on $K$, with $k = \max\{k_i : 1 \leq i \leq N\}$. Define $S : K \to H$ by $Sx = kx + (1 - k)Tx$.

By proposition 2.9, we know that $S : K \to H$ is nonexpansive and $F(S) = F(T)$. By our assumption on $T$, we know $F(T) \neq \emptyset$ and hence $F(S) \neq \emptyset$.

Since $S : K \to H$ is nonexpansive, then $S : K \to H$ is $k$-strict pseudo-contraction on $K$, where $k = 0$. By proposition 2.8, we see that $F(P_K S) = F(S) \neq \emptyset$.

Since $P_K : H \to K$ is nonexpansive, we conclude that $P_K S : K \to K$ is nonexpansive.
From the control condition on $\{\alpha_n\}$, we have

$$\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \frac{1}{(1 - k)^2} \sum_{n=1}^{\infty} (\alpha_n - k)(1 - \alpha_n) = \infty.$$  

Then, by Theorem 2 given by Reich in [7] to deduce that $\{x_n\}$ converges weakly to a fixed point of $P_K S$.

Notice that $F(P_K S) = F(S) = F(T)$, we have the conclusion.

The proof is completed.

From Theorem 3.1, we can deduce Theorem 3.2 of Marino and Xu [24].

**Corollary 3.2.** (Xu [24]) Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $T_i : K \to K$, be a $k_i$-strict pseudo-contraction for some $0 \leq k_i < 1$. Let $k = \max\{k_i : 1 \leq i \leq N\}$. Assume the common fixed point set $\bigcap_{i=1}^{N} Fix(T_i)$ is nonempty. Assume also $\{\lambda_i\}_{i=1}^{N}$ is a finite sequence of positive numbers, such that $\sum_{i=1}^{N} \lambda_i = 1$. Given $x_0 \in K$, let $\{x_n\}_0^{\infty}$ be the sequence generated by Mann’s algorithm:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \sum_{i=1}^{N} \lambda_i T_i x_n$$

Assume the control sequence $\{\alpha_n\}_0^{\infty}$ is chosen so that $k < \alpha_n < 1$ for all $n$ and

$$\sum_{n=1}^{\infty} (\alpha_n - k)(1 - \alpha_n) = \infty.$$  

Then $\{x_n\}$ converges weakly to a common fixed point $\{T_i\}_1^{N}$.

**Proof.** We observe first that, for all $x \in K$.

$$P_K[kI + (1 - k) \sum_{i=1}^{N} \lambda_i T_i] x = [kI + (1 - k) \sum_{i=1}^{N} \lambda_i T_i] x$$

Since $T_i : K \to K$, thus $kI + (1 - k) \sum_{i=1}^{N} \lambda_i T_i : K \to K$ is a self-mapping.

For given $\{\alpha_n\}$, by the choice of $\{\beta_n\}$, we get

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \sum_{i=1}^{N} \lambda_i T_i x_n$$

$$= [k + (1 - k) \beta_n] x_n + (1 - k)(1 - \beta_n) \sum_{i=1}^{N} \lambda_i T_i x_n$$

$$= \beta_n x_n + (1 - \beta_n)[k x_n + (1 - k) \sum_{i=1}^{N} \lambda_i T_i x_n]$$

$$= \beta_n x_n + (1 - \beta_n) P_K[k x_n + (1 - k) \sum_{i=1}^{N} \lambda_i T_i x_n]$$

Consequently, we conclude that $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_1^{N}$ by Theorem 3.1.

The proof is completed.

**Remark 3.3.** Theorem 3.1 and its Corollary mainly improves Xu [24] in the following senses:

(i) relaxing the restriction on $\{\alpha_n\}$ from $(k, 1)$ to $[k, 1]$;

(ii) from $k$-strict pseudo-contraction self-mapping to non-self mapping.
In order to get a strong convergence theorem, we modify the iterative algorithm for $k$-strict pseudo-contraction. We have the following theorem.

**Theorem 3.4.** Let $K$ be a nonempty closed convex subset of a Hilbert space $H$ and $T_i: K \to H$ be a $k_i$-strictly pseudo-contractive nonself-mapping, for some $0 \leq k_i < 1$, let $k = \max \{k_i : 1 \leq i \leq N\}$. Assume the common fixed point set $\bigcap_{i=1}^{N} \text{Fix}(T_i)$ is nonempty. Assume also for each $n$, $\{\lambda_i\}_{i=1}^{N}$ is a finite sequence of positive numbers, such that $\sum_{i=1}^{N} \lambda_i = 1$ for all $1 \leq i \leq N$. Given $u \in K$ and sequences $\{\alpha'_n\}$ and $\{\beta'_n\}$ in $(0,1)$, satisfying control conditions: (i) $\sum_{n=1}^{\infty} \beta'_n = \infty$; $\beta'_n \to 0$, (ii) $k \leq \alpha'_n \leq b < 1$ for all $n \geq 1$, and (iii) $\sum_{n=1}^{\infty} |\alpha'_{n+1} - \alpha'_n| < \infty$, $\sum_{n=1}^{\infty} |\beta'_{n+1} - \beta'_n| < \infty$, or $\frac{\beta'_n}{\beta'_{n+1}} \to 1$ as $n \to \infty$, let the sequence $\{x'_n\}$ be generated by (1.8), i.e.,

$$x'_{n+1} = \beta'_n u + (1 - \beta'_n)P_K[\alpha'_n x'_n + (1 - \alpha'_n) \sum_{i=1}^{N} \lambda_i T_i x'_n]$$

Then, $\{x'_n\}$ converges strongly to a common fixed point $z$ of $\{T_i\}_{i=1}^{N}$, where $z = P_{F(T)}u$ and $T = \sum_{i=1}^{N} \lambda_i T_i$.

**Proof.** 1. $\{x'_n\}$ is bounded. By Proposition 2.5, we know that $\text{Fix}(\sum_{i=1}^{N} \lambda_i T_i) = \bigcap_{i=1}^{N} \text{Fix}(T_i) \neq \emptyset$, take $p \in \bigcap_{i=1}^{N} \text{Fix}(T_i)$, from (1.8), we have

$$\|x'_{n+1} - p\| \leq \beta'_n \|u - p\| + (1 - \beta'_n) \|P_K[\alpha'_n x'_n + (1 - \alpha'_n) T x'_n] - p\|$$

$$\leq \beta'_n \|u - p\| + (1 - \beta'_n) \|\alpha'_n x'_n + (1 - \alpha'_n) T x'_n - p\|^2$$

$$= \beta'_n \|u - p\| + (1 - \beta'_n) [\alpha'_n \|x'_n - p\|^2 + (1 - \alpha'_n) \|T x'_n - p\|^2 - \alpha'_n (1 - \alpha'_n) \|x'_n - T x'_n\|^2]$$

$$= \beta'_n \|u - p\| + (1 - \beta'_n) [\|x'_n - p\|^2 - (1 - \alpha'_n) (\alpha'_n - k) \|x'_n - T x'_n\|^2]$$

$$\leq \beta'_n \|u - p\| + (1 - \beta'_n) \|x'_n - p\|^2 \leq \max\{\|u - p\|, \|x'_n - p\|\} \leq \max\{\|u - p\|, \|x'_n - p\|\} \leq \max\{\|u - p\|, \|x'_n - p\|\}$$

By induction, $\|x'_{n+1} - p\| \leq \max\{\|u - p\|, \|x'_1 - p\|\}$, $n \geq 0$, i.e., $\{x'_n\}$ is bounded.

2. $\limsup_{n \to \infty} \langle u - P_{F(T)}u, y'_n - P_{F(T)}u \rangle \leq 0$.

By Proposition 2.5, we also have $T$ is a $k$-strictly pseudo-contraction on $K$ with $k = \max\{k_i : 1 \leq i \leq N\}$. Proposition 2.6 ensures that $P_{F(T)}u$ is well defined.

$P_K[\alpha'_n I + (1 - \alpha'_n) T] : K \to K$ is a nonexpansive mapping. Indeed, by using Lemma 2.1, the definition of strictly pseudocontraction and condition (ii), we
have for all \( x, y \in K \) that

\[
\| P_K[\alpha'_n I + (1 - \alpha'_n)T]x - P_K[\alpha'_n I + (1 - \alpha'_n)T]y \| \leq \|\alpha'_n(x - y) + (1 - \alpha'_n)(Tx - Ty)\| \\
= \alpha'_n \|x - y\|^2 + (1 - \alpha'_n)\|Tx - Ty\|^2 \\
- \alpha'_n(1 - \alpha'_n)\|x - Tx - (y - Ty)\|^2 \\
\leq \alpha'_n \|x - y\|^2 + (1 - \alpha'_n)\|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 \\
- \alpha'_n(1 - \alpha'_n)\|x - Tx - (y - Ty)\|^2 \\
= \|x - y\|^2 - (1 - \alpha'_n)(\alpha'_n - k)\|x - Tx - (y - Ty)\|^2 \\
\leq \|x - y\|^2
\]

which imply that \( P_K[\alpha'_n I + (1 - \alpha'_n)T] \) is nonexpansive.

Next we prove that \( \|x'_{n+1} - x'_n\| \to 0 \) as \( n \to \infty \).

To this end, we first estimate \( \|y'_n - y'_{n-1}\| \). Set \( M_1 = \sup\{\|x'_n - Tx'_{n-1}\|\} \) and \( M_2 = \|u\| + \sup\{\|y'_n\|\} \), then, by (1.8) and noting that \( P_K[\alpha'_n I + (1 - \alpha'_n)T] \) is nonexpansive, we have

\[
\|y'_n - y'_{n-1}\| = \| P_K[\alpha'_n I + (1 - \alpha'_n)T]x'_n - P_K[\alpha'_{n-1} I + (1 - \alpha'_{n-1})T]x'_{n-1} \| \\
\leq \| P_K[\alpha'_n I + (1 - \alpha'_n)T]x'_n - P_K[\alpha'_n I + (1 - \alpha'_n)T]x'_{n-1} \| \\
+ P_K[\alpha'_n I + (1 - \alpha'_n)T]x'_{n-1} - P_K[\alpha'_{n-1} I + (1 - \alpha'_{n-1})T]x'_{n-1} \| \\
\leq \|x'_n - x'_{n-1}\| + \| P_K[\alpha'_n I + (1 - \alpha'_n)T]x'_{n-1} - P_K[\alpha'_{n-1} I + (1 - \alpha'_{n-1})T]x'_{n-1} \| \\
\leq \|x'_n - x'_{n-1}\| + M_1|\alpha'_n - \alpha'_{n-1}| \\
\]

(3.1)

then, from (3.1), we get

\[
\|x'_{n+1} - x'_n\| \leq \|(1 - \beta'_n)\|y'_n - y'_{n-1}\| + M_2|\beta'_n - \beta'_{n-1}| \\
\leq \|(1 - \beta'_n)(\|x'_n - x'_{n-1}\| + M_1|\alpha'_n - \alpha'_{n-1}|) + M_2|\beta'_n - \beta'_{n-1}| \)

(3.2)

By Lemma 2.4, we conclude that \( \|x'_n - x'_{n-1}\| \to 0 \) as \( n \to \infty \).

Noting that \( \|x'_{n+1} - y'_n\| = \beta'_n\|u - y'_n\| \to 0 \) as \( n \to \infty \), combining this and (3.2), we have \( \|x'_n - y'_n\| \to 0 \) as \( n \to \infty \).

On the other hand, by condition (ii) and (iii), we have \( \alpha'_n \to \alpha \) as \( n \to \infty \), where \( \alpha \in [k, 1) \). Define \( S : K \to H \) by \( Sx = \alpha x + (1 - \alpha)Tx \).

Then, \( S \) is nonexpansive mapping with \( F(S) = F(T) \) by proposition 2.9, it follows from proposition 2.7 that \( F(P_K S) = F(S) = F(T) \).

Set \( M_3 = \sup\{\|x'_n\| + \|Tx'_n\| : n \geq 1\} \). Since

\[
\|P_K Sx'_n - y'_n\| \leq M_3|\alpha'_n - \alpha'_{n-1}| \to 0, \ \text{as} \ n \to \infty,
\]
then we have
\[ \|x_n' - P_KSx_n'\| \leq \|x_n' - y_n'\| + \|y_n' - P_KSx_n'\| \to 0, \text{ as } n \to \infty. \]

We now prove that \( \limsup_{n \to \infty} \langle u - P_{F(T)}u, y'_n - P_{F(T)}u \rangle \leq 0 \).

To see this, assume that
\[ \limsup_{n \to \infty} \langle u - P_{F(T)}u, y'_n - P_{F(T)}u \rangle = \lim_{j \to \infty} \langle u - P_{F(T)}u, y'_{nj} - P_{F(T)}u \rangle. \]

Without loss of generality, assume that \( y'_{nj} \rightharpoonup p \) as \( j \to \infty \),
then \( x'_{nj} \rightharpoonup p \) and \( \|x'_{nj} - P_KSx'_{nj}\| \to 0 \) as \( j \to \infty \).

By Lemma 2.2 we have \( p \in F(P_KS) = F(T) \).

By lemma 2.3, we have that
\[ \langle u - P_{F(T)}u, p - P_{F(T)}u \rangle \leq 0. \]

Hence,
\[ \limsup_{n \to \infty} \langle u - P_{F(T)}u, y'_n - P_{F(T)}u \rangle \leq 0. \]

3. we prove that \( x'_n \to P_{F(T)}u \) as \( n \to \infty \).

Putting \( \gamma_n = \max \{ \langle u - P_{F(T)}u, y'_n - P_{F(T)}u \rangle, 0 \} \), then \( \gamma_n \to 0 \) as \( n \to \infty \).

By lemma 2.1, we have
\[ \|x'_{n+1} - P_{F(T)}u\|^2 = (1 - \beta'_n)^2\|y'_n - P_{F(T)}u\|^2 + \beta'_n^2\|u - P_{F(T)}u\|^2 + 2\beta'_n(1 - \beta'_n)\langle u - P_{F(T)}u, y'_n - P_{F(T)}u \rangle \leq (1 - \beta'_n)\|x'_n - P_{F(T)}u\|^2 + o(\beta'_n) \]
which leads to \( x'_n \to P_{F(T)}u \) as \( n \to \infty \), by virtue of lemma 2.4.

This completes the proof.

References

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