Moment Generating Function of the Bivariate Generalized Exponential Distribution

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Abstract

Recently a new distribution, named a bivariate generalized exponential (BVGE) distribution has been introduced by Kundu and Gupta (2008). In this paper we obtain joint moments and the moment generating function for (BVGE) which is in closed form, and convenient to use in practice.

Keywords: bivariate generalized exponential distribution; joint and marginal moments; moment generating function; marginal moment generating function.

1. Introduction

The two parameter generalized exponential distribution was introduced by Gupta and Kundu (1999). Gupta and Kundu (2001a) observed that it can be used quite effectively to analyze positive life time data, particularly, in place of the two-parameter gamma or two-parameter Weibull distributions. Since the distribution function of the generalized exponential is in closed form, it can be used quite easily for analyzing censored data also. The frequentest and Bayesian inferences have been developed for the unknown parameters of the generalized exponential distribution. The readers are referred to the review article of Gupta and Kundu (2007) for a current account on generalized exponential distribution.

Although quite bit of work has been done in the recent years on generalized exponential distribution, but not much attempt has made to extend this to the multivariate case. Recently Kundu and Gupta (2008) define a bivariate generalized exponential distribution (BVGE) distribution as: The bivariate vector \((X_1, X_2)\) has a bivariate generalized exponential distribution with the shape parameters \(\alpha_1, \alpha_2\) and \(\alpha_3\), denoted by \(BGE(\alpha_1, \alpha_2, \alpha_3)\). if \((X_1, X_2)\) has the joint probability density function

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2. Joint and Marginal Moments

In this section we drive the \( r \)th and \( s \)th joint moments of \( X_1 \) and \( X_2 \) as well as the marginal moments.

**Theorem 2.1.** The \( r \)th and \( s \)th joint moments of the bivariate generalized exponential distribution, denoted by \( \mu'_{r,s} \), is given by

\[
\mu'_{r,s} = \text{E}(X_1^r, X_2^s) = \alpha_2 (\alpha_1 + \alpha_3) ! \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{k_{ij}}{(j+1)^{r+s+1}} \left[ \frac{\Gamma(s+1)}{(i+1)^{s+1}} - \sum_{k=0}^{\infty} \frac{\Gamma(s+k+1)}{k!(i+2)^{s+k+1}} \right] \\
+ \alpha_1 (\alpha_2 + \alpha_3) ! \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{k_{ij}'}{(j+1)^{r+s+1}} \left[ \frac{\Gamma(r+1)}{(i+1)^{r+1}} - \sum_{k=0}^{\infty} \frac{\Gamma(r+k+1)}{k!(i+2)^{r+k+1}} \right] \\
+ \alpha_3 \sum_{i=0}^{\infty} \frac{k_{ii}}{(i+1)^{r+s+1}} \Gamma(r+s+1), \quad r,s = 1,2,3,\ldots
\]

Where
Moment generating function

\[ k_{ij} = (-1)^{i+j} \binom{\alpha_2 - 1}{i} \binom{\alpha_1 + \alpha_3 - 1}{j}, \quad k'_{ij} = (-1)^{i+j} \binom{\alpha_1 - 1}{i} \binom{\alpha_2 + \alpha_3 - 1}{j}, \]

\[ k_i = (-1)^i \binom{\alpha_1 + \alpha_2 + \alpha_3 - 1}{i}, \]

and

\[ \Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \, du, \quad x > 0 \]

is the complete gamma function.

**Proof.** Starting with

\[ \text{E}(X_1^r, X_2^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s f(x_1, x_2) \, dx_1 \, dx_2, \quad r, s = 1, 2, 3, \ldots \]

and substituting for \( f(x_1, x_2) \) from (1.1), we get

\[ \text{E}(X_1^r, X_2^s) = \alpha_2 (\alpha_1 + \alpha_3) \int_0^\infty \int_0^\infty x_1^r x_2^s (1 - e^{-x_1})^{\alpha_1 + \alpha_3 - 1} (1 - e^{-x_2})^{\alpha_2 - 1} e^{-x_1 - x_2} \, dx_1 \, dx_2 \]

\[ + \alpha_1 (\alpha_2 + \alpha_3) \int_0^\infty \int_0^\infty x_1^r x_2^s (1 - e^{-x_2})^{\alpha_2 + \alpha_3 - 1} (1 - e^{-x_1})^{\alpha_1 - 1} e^{-x_1 - x_2} \, dx_2 \, dx_1 \]

\[ + \alpha_3 \int_0^\infty x_2^{r+s} (1 - e^{-x_2})^{\alpha_1 + \alpha_2 + \alpha_3 - 1} e^{-x_2} \, dx \]

upon using the binomial series expansion

\[ (1 - e^{-x})^{\alpha - 1} = \sum_{i=1}^{\infty} (-1)^i \binom{\alpha - 1}{i} e^{-ix} \]

and the fact that

\[ \gamma(c, z) = \int_0^z u^{c-1} e^{-u} \, du = \Gamma(c)[1 - e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{k!}]; \quad c \in \mathbb{N}^* \quad (2.1) \]

where \( \gamma(c, z) \) is an incomplete gamma function,

The expression for \( \text{E}(X_1^r, X_2^s) \) is derived.

**Comment (2.1).** From theorem 2.1, it easily to obtain different marginal moments of \( X_1 \) and \( X_2 \) in terms of infinite series by putting \( r = 0 \) and \( s = 0 \) respectively, that is the rth marginal moment of \( X_1 \) will be
\[ \mu_{r0} = E(X_1^r) = \alpha_2 (\alpha_1 + \alpha_3) r! \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{k_{ij}}{(j+1)^{r+1}} \left[ \frac{1}{(i+1)} - \sum_{k=0}^{r} \frac{1}{(i+2)^{k+1}} \right] + \alpha_1 (\alpha_2 + \alpha_3) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{k'_{ij}}{(j+1)^{r+1}} \left[ \frac{\Gamma(r+1)}{(i+1)^{r+1}} - \frac{\Gamma(r+1)}{(i+2)^{r+1}} \right] + \alpha_3 \sum_{i=0}^{\infty} k_i \frac{\Gamma(r+1)}{(i+1)^{r+1}}. \]

Similarly, the sth marginal moment of \( X_2 \) is given by
\[ \mu'_{s0} = E(X_2^s) = \alpha_2 (\alpha_1 + \alpha_3) s! \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{k_{ij}}{(j+1)^{s+1}} \left[ \frac{1}{(i+1)} - \sum_{k=0}^{s} \frac{1}{(i+2)^{k+1}} \right] + \alpha_1 (\alpha_2 + \alpha_3) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{k'_{ij}}{(j+1)^{s+1}} \left[ \frac{\Gamma(s+1)}{(i+1)^{s+1}} - \frac{\Gamma(s+1)}{(i+2)^{s+1}} \right] + \alpha_3 \sum_{i=0}^{\infty} k_i \frac{\Gamma(s+1)}{(i+1)^{s+1}}. \]

We observe that the infinite series is summable and it has only a finite number of terms if the shape parameters are integers.

### 3. Joint Moment Generating Function

The following theorem gives the joint moment generating function of \((X_1, X_2)\).

**Theorem 3.1 :** The moment generating function for the BVGED is given by

\[
M(t_1,t_2) = \alpha_2 B(1-t_2, \alpha_1 + \alpha_2 + \alpha_3) \times \\
\times F_2[1-t_2, \alpha_1 + \alpha_3, t_1; \alpha_1 + \alpha_3 + 1, \alpha_1 + \alpha_2 + \alpha_3 + 1 - t_2; 1] + \alpha_1 B(1-t_1, \alpha_1 + \alpha_2 + \alpha_3) \times \times F_2[1-t_1, \alpha_2 + \alpha_3, t_2; \alpha_2 + \alpha_3 + 1, \alpha_1 + \alpha_2 + \alpha_3 + 1 - t_1; 1] + \alpha_3 B(1-t_1 + t_2, \alpha_1 + \alpha_2 + \alpha_3), \tag{3.1}
\]

where
\[
B(\alpha, \beta) = \int_0^{\infty} u^{\alpha-1} (1-u)^{\beta-1} \, du, \quad \text{is a beta function,}
\]
Moment generating function

\[ pFq(b_1, \ldots, b_p; c_1, \ldots, c_q; u) = \sum_{i=0}^{\infty} \frac{(b_1)_i \cdots (b_p)_i}{(c_1)_i \cdots (c_q)_i} \frac{u^i}{i!}, \]

is a hypergeometric function,

and \( (b)_i = (b + 1) \cdots (b + i - 1) \)

\( \frac{\Gamma(b + i)}{\Gamma(b)} (b \neq 0, i = 1, 2, \ldots) \), and \( p, q \) are nonnegative integers.

**Proof:** The joint moment generating function of \((X_1, X_2)\) is given by

\[ M(t_1, t_2) = E[e^{t_1 X_1 + t_2 X_2}] = \int_0^\infty e^{t_1 x_1 + t_2 x_2} f(x_1, x_2) \, dx_1 \, dx_2 \]

substituting for \( f(x_1, x_2) \) from (1.1) we get

\[
M(t_1, t_2) = \alpha_2 (\alpha_1 + \alpha_3) \int_0^\infty \int_0^\infty (1 - e^{-x_2})^{\alpha_2 - 1} e^{-(t_2 - t_1)x_2} \int_0^\infty \int_0^\infty (1 - e^{-x_1})^{\alpha_1 - 1} e^{-(t_1 - t_2)x_1} \, dx_2 \, dx_1 \\
+ \alpha_1 (\alpha_2 + \alpha_3) \int_0^\infty \int_0^\infty (1 - e^{-x_1})^{\alpha_1 - 1} e^{-(t_1 - t_2)x_1} \int_0^\infty \int_0^\infty (1 - e^{-x_2})^{\alpha_2 - 1} e^{-(t_2 - t_1)x_2} \, dx_1 \, dx_2 \\
+ \alpha_3 \int_0^\infty e^{(t_1 + t_2)x} (1 - e^{-x})^{\alpha_1 + \alpha_2 + \alpha_3 - 1} \, dx. \tag{3.2}
\]

Upon using the fact that

\[ B_\alpha(\alpha, \beta) = \int_0^x u^{\alpha-1} (1-u)^{\beta-1} \, du = \frac{x^\alpha}{\alpha} \text{B}((\alpha, 1 - \beta; \alpha + 1; x), \quad (0 \leq x \leq 1) \]

where \( B_\alpha(\alpha, \beta) \) is an incomplete beta function

and the identity [see Sarhan and Balakrishnan (2007)]

\[ \int_0^\infty u^{-\alpha}(1-u)^{\beta-1} \, du = B(\alpha, \beta) \]

\[ \text{F}_2(\alpha, \beta; c, d; \rho; u) \]

for \( \alpha, \beta > 0 \) and \( d + \beta - \alpha - c > 0 \),

we can drive the expression for \( M(t_1, t_2) \) given in (3.1).

To show that \( M(0,0) = 1 \), set \( t_1 = t_2 = 0 \) in (3.1)

Since

\[ \text{F}_2[1, \alpha_1 + \alpha_3 + 0, \alpha_1 + \alpha_3 + 1, \alpha_1 + \alpha_2 + \alpha_3 + 1; 1] = 1 \], \quad i = 1, 2

then

\[ M(0,0) = (\alpha_1 + \alpha_2 + \alpha_3) \cdot B[1, \alpha_1 + \alpha_2 + \alpha_3] = 1. \]

Another form for the moment generating function can also be obtained in the form of a series which is finite or infinite depending on whether the shape parameters is integers or not. By using the binomial series expansion , (3.2) can be written as
Since the quantity inside the summation is absolutely integrable, interchanging the summation and integration, and by using (2.1) we have

\[
M(t_1, t_2) = \alpha_2 (\alpha_1 + \alpha_3) \sum_{i=0}^{\infty} (-1)^i \left( \begin{array}{c} \alpha_2 - 1 \\ i \end{array} \right) e^{-x_2(1-t_1+i)}
\]
\[
\times \int \sum_{j=0}^{\infty} (-1)^j \left( \begin{array}{c} \alpha_1 + \alpha_3 - 1 \\ j \end{array} \right) e^{-x_1(1-t_i+i)} \, dx_1 \, dx_2
\]
\[
+ \alpha_1 (\alpha_2 + \alpha_3) \sum_{i=0}^{\infty} (-1)^i \left( \begin{array}{c} \alpha_1 - 1 \\ i \end{array} \right) e^{-x_1(1-t_i+i)}
\]
\[
\times \int \sum_{j=0}^{\infty} (-1)^j \left( \begin{array}{c} \alpha_2 + \alpha_3 - 1 \\ j \end{array} \right) e^{-x_2(1-t_i+t_j)} \, dx_2 \, dx_1
\]
\[
+ \alpha_3 \sum_{i=0}^{\infty} (-1)^i \left( \begin{array}{c} \alpha_1 + \alpha_2 + \alpha_3 - 1 \\ i \end{array} \right) e^{-x(1-t_i+t_j)} \, dx
\]

We observe that the infinite series is summable, differentiable and it has only a finite number of terms if the shape parameters are integers. Hence we can obtain different joint moments by differentiating and evaluating at \( t_1 = t_2 = 0 \).

**Comment 3.1.** From equation (3.3), it is easy to obtain the marginal moment generating functions of \( X_1 \) and \( X_2 \) in terms of infinite series by putting \( t_2 = 0 \) and \( t_1 = 0 \) respectively, that is the marginal moment generating function of \( X_1 \) will be
Moment generating function

\[
M(t_1,0) = \alpha_2 (\alpha_1 + \alpha_3) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} k_{ij} \left( \frac{1}{1-t_1 + j} \right) \left[ \frac{1}{1+i} - \frac{1}{2+i} \right] \\
+ \alpha_1 (\alpha_2 + \alpha_3) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} k'_{ij} \left( \frac{1}{1-t_1 + i} \right) \left[ \frac{1}{1-j-1} - \frac{1}{2-t_1 + i} \right] \\
+ \alpha_3 \sum_{i=0}^{\infty} k_i \left( \frac{1}{1-t_1 + i} \right).
\]

Similarly, the marginal moment generating function of \( X_2 \) is given by

\[
M(0,t_2) = \alpha_2 (\alpha_1 + \alpha_3) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} k_{ij} \left( \frac{1}{1+t_2 + j} \right) \left[ \frac{1}{1-i} - \frac{1}{2-t_2 + i} \right] \\
+ \alpha_1 (\alpha_2 + \alpha_3) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} k'_{ij} \left( \frac{1}{1-t_2 + j} \right) \left[ \frac{1}{1+i} - \frac{1}{2+i} \right] \\
+ \alpha_3 \sum_{i=0}^{\infty} k_i \left( \frac{1}{1-t_2 + i} \right).
\]

References


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