Applied Mathematical Sciences, Vol. 3, 2009, no. 55, 2707 - 2724

The Number of Certain Rankings and Hierarchies Formed from Labeled or Unlabeled Elements and Sets

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Abstract

Certain types of combinatorial rankings and hierarchies of labeled or unlabeled elements are presented along with their counting formulas.

Mathematics Subject Classification: 05A15, 05A18

Keywords: enumerative combinatorics, rankings, hierarchies

1 Introduction

Rankings and hierarchies are well-known structures in discrete mathematics. In this paper, we will give a synopsis over counting formulas for certain rankings and hierarchies formed from unlabeled or labeled elements. Furthermore, these structures may be dissected also, as first described in [9].

2 Counting formulas

Consider a certain structure like the well-known preferred arrangement. This structure is the most prominent example for a ranking and is defined as "Number of ways n competitors can rank in a competition, allowing for the possibility of ties", see A000670 in [7]. Thus, for given n we have a certain number $a(n)$ of realizations of the given structure. This $a(n)$ is an integer sequence. For each structure described in this paper we will give $a(n)$. Several methods are possible to find $a(n)$. We briefly mention that $a(n)$ can be found by 1) generating all realizations for given n (by hand or by aid of an computer program), 2) by the method of generating functions [10, 1], 3) by derivation of an explicit or a recursive counting formula [10, 1]. We will use $a(n)$ for the sequence itself and for its counting formula(s). Most of the counting formulas from sections 5 and 6 along with further comments can be found in the On-Line Encyclopedia of Integer Sequences (OEIS) [7, 8].

3 Description of rankings and hierarchies

We consider either a set $\mathcal{U} = \{1, 1, 1, \ldots\}$ of n unlabeled elements or a set $\mathcal{L} = \{1, 2, 3, \ldots, n\}$ of n labeled elements. Furthermore, we consider a list of n levels $\mathcal{A} = [l_1, l_2, l_3, \ldots, l_n]$ in natural, that is in consecutive order. We may call $\mathcal{A}$ a ladder and instead of levels we sometimes speak of ranks, too. Now we can distribute the elements of either $\mathcal{U}$ or $\mathcal{L}$ onto the levels of $\mathcal{A}$. We do not mix labeled and unlabeled elements and we do not allow for empty levels. The occupation number $o(l_j)$ of level l_j tells us how many elements we have on the j-th level. If $o(l_j)$ is a monotonic decreasing function of j, then we have a hierarchy. If not, then we have a ranking. Please note that for hierarchies we do not request $o(l_j)$ to be strictly monotonic, thus $o(l_j) = o(l_{j+1})$ is allowed. We will speak of structures if we mean both rankings and hierarchies.

Now a further concept comes into play. We may at first partition the set $\mathcal{U}$ into m subsets $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m\}$. The same can be done on $\mathcal{L}$. Subsequently we can form rankings or hierarchies on the subsets as described above. We get a set of structures which we will call dissected rankings or dissected hierarchies. Two of those structures have been described earlier [9]. Structures without dissection are named single structures when necessary.

The possible combinations of the three categories 1) unlabeled/labeled elements, 2) ranking/hierarchy, and 3) single/multiple structures result in four rankings and four hierarchies. Examples for each structure are given in table 1. In any case, we shall not mix different types of rankings or hierarchies into a dissected structure.

Finally, for another type of structures we replace elements by sets. Thus we form subsets from $\mathcal{U}$ or $\mathcal{L}$ and then we distribute these subsets onto $\mathcal{A}$. Again, we can build single structures and multiple, i.e. dissected structures. Examples for the possible four rankings and hierarchies build from sets are displayed in table 4.

4 Description of the tables

The tables given below have the following composition in common. Rankings are placed in the first half of the table, hierarchies in the second half. The left side concerns structures for unlabeled elements, the right side the corresponding structures for labeled elements. Below each single structure the corresponding dissected structure is mentioned.

In order to display a structure (see tables 1 and 4) we have the following conventions: The comma separates two elements or sets, e.g. 1,2,3 are just three elements. The semicolon separates two allowed structures, e.g. 1,2,3; 1 : 2 : 3 are two possible hierarchies. The colon separates two levels, e.g. in 1,2,3 : 4,5 : 6 we have a hierarchy composed of six elements distributed onto

Table 1: Examples for rankings and hierarchies among elements.

	unlabeled	labeled
Ranking	A000079: 1,1,1; 1,1 : 1; 1 : 1,1; 1 : 1 : 1	A000670: 1,2,3; 1,2 : 3; 1 : 2,3; 1 : 2 : 3; 1,3 : 2; 1 : 3,2; 1 : 3 : 2; 2,3 : 1; 2 : 3,1; 2 : 3 : 1; 2 : 1 : 3; 3 : 1 : 2; 3 : 2 : 1
Dissected ranking	A034691: 1,1,1; 1,1 : 1; 1 : 1,1; 1 : 1 : 1; 1,1|1; 1 : 1|1; 1|1|1	A075729: 1,2,3; 1 : 2,3; 3 : 1,2; 2 : 1,3; 3,1 : 2; 2,3 : 1; 1,2 : 3; 1 : 2 : 3; 3 : 1 : 2; 2 : 3 : 1; 3 : 2 : 1; 2 : 1 : 3; 1 : 3 : 2; 1,2|3; 3,1|2; 2,3|1; 1 : 2|3; 3 : 1|2; 2 : 3|1; 3 : 2|1; 1 : 3|2; 2 : 1|3; 1|3|2
Hierarchy	A000041: 1,1,1; 1,1 : 1; 1 : 1 : 1	A005651: 1,2,3; 1,2 : 3; 1 : 2 : 3; 1,3 : 2; 3 : 1 : 2; 2,3 : 1; 2 : 3 : 1; 1 : 3 : 2; 2 : 1 : 3; 3 : 2 : 1
Dissected hierarchy	A001970: 1,1,1; 1,1 : 1; 1 : 1 : 1; 1,1|1; 1 : 1|1; 1|1|1	A143463: 1,2,3; 1,2 : 3; 1 : 2 : 3; 1,2|3; 1 : 2|3; 1|2|3; 1,3 : 2; 3 : 1 : 2; 1,3|2; 1 : 3|2; 2,3 : 1; 2 : 3 : 1; 2,3|1; 2 : 3|1; 1 : 3 : 2; 2 : 1|3; 2 : 1 : 3; 3 : 1|2; 3 : 2 : 1; 3 : 2|1

three levels. The first three elements 1,2,3 are on level 1, the element 6 resides on level 3. The line | separates two substructures, e.g. 1 : 2 | 3,4 : 5 tells us that we have a dissected hierarchy composed of the two hierarchies 1 : 2 and 3,4 : 5.

Tables 2 and 5 constitute the main part of this paper, because they contain the integer sequences and their counting formulas. For each structure the following items are given, if known: the integer sequence $a(n)$ and its OEIS number, the name of $a(n)$, transformation relations of $a(n)$ into other integer sequences, the generating function of $a(n)$, explicit, recursive and asymptotic counting formulas for $a(n)$, and a comstruct command [3] to explicitly generate the structure.

5 Rankings and Hierarchies for Elements

Table 1 gives examples for rankings and hierarchies build from unlabeled or labeled elements. In total eight examples are provided according to the four types of rankings and the four types of hierarchies as introduced in section 3.

Table 2 assembles the counting formulas for the structures from table 1. We do not give derivations for all these formulas. Many of the formulas are simple and known since long. The authorships of the formulas can be found in the OEIS using the corresponding OEIS numbers which are all given here.

Elements	
unlabeled	**labeled**

Rankings	
A000079 = 1, 2, 4, 8, 16, 32, 64, ...	A000670 = 1, 3, 13, 75, 541, 4683, ...
Powers of 2 = Number of subsets of an n-set.	Preferential arrangements of n labeled elements.
Invert transform of A000012 = 1, 1, 1,...	First Eulerian transform of A000079, Stirling transform of A000142 = $k!$
G.f.: $\frac{1}{1-2x}$, E.g.f.: $\exp(2x)$.	E.g.f.: $\frac{1}{2-\exp(x)}$.
$a(n) = 2^n$.	$a(n) = \sum_{k=1}^{n} k!\ S_2(n,k)$.
$a(n) = \sum_{k=1}^{n} a(n-k)A000012(k)$.	$a(n) = \sum_{k=1}^{n} C(n,k)\ a(n-k)$.
$a(n) = \sum_{i=1}^{P(n)} \frac{T(i,n)!}{\prod_{s=1}^{D(i,n)} m(i,s)!}$.	$a(n) = \sum_{i=1}^{P(n)} \frac{n!}{\prod_{j=1}^{T(i,n)} t(i,j)!} \frac{T(i,n)!}{\prod_{s=1}^{D(i,n)} m(i,s)!}$.
$a(n) \asymp \exp(-n)^{-\ln(2)}$	$a(n) \asymp \frac{1}{2}\ n!\ \log_2(e)^{(n+1)}$, where $\log_2(e) = 1.442695$.
SeqSetU := [S, { S = Sequence(U), U = Set(Z, card $\geq$ 1)}, unlabeled];	SeqSetL := [S, { S = Sequence(U), U = Set(Z, card $\geq$ 1)}, labeled];

Dissected Rankings	
A034691 = 1, 3, 7, 18, 42, 104, 244, ...	A075729 = 1, 4, 23, 173, 1602, 17575, ...
Number of different hierarchical orderings that can be formed from n unlabeled elements.	Number of different hierarchical orderings that can be formed from n labeled elements.
Euler transform of A000079.	Exp transform of A000670, Stirling transform of A000262.

G.f.: $1+\sum_{n=1}^{\inf} a(n)x^n = \frac{1}{\prod_{n=1}^{\inf}(1-x^n)^{2^{n-1}}}$.

$a(n) = \frac{1}{n}\sum_{k=1}^{n-1} c(k)\ A000079(n-k)$, with $c(n) = \sum_{k|n} k\ A000079(k)$.

$a(n) = \sum_{i=1}^{P(n)} \prod_{s=1}^{D(i,n)} C(2^{d(i,s)-1} + m(i,s) - 1, m(i,s))$.

$a(n) \asymp \frac{2^n \exp\left(\sqrt{2n}\right)}{\sqrt{2\pi} 2^{\frac{3}{4}} e^{\frac{3}{4}} n^{\frac{1}{4}}}$

SetSeqSetU := [T, {T = Set(S), S = Sequence(U, card ≥ 1), U = Set(Z, card ≥ 1)}, unlabeled];

E.g.f.: $\exp\left(\frac{1}{2-\exp(x)} - 1\right)$.

$a(n) = \sum_{k=1}^{n} C(n-1, k-1)\ a(n-k)\ A000670(k)$.

$a(n) = \sum_{k=1}^{n} A000262(n)\ S_2(n,k)$.

$a(n) = \sum_{i=1}^{P(n)} \frac{n!\ \prod_{s=1}^{D(i,n)} B(d(i,j))^{m(i,s)}}{\prod_{s=1}^{D(i,n)} m(i,s)!\ (d(i,s)!)^{m(i,s)}}$.

$a(n) \asymp \frac{n! \exp\left(\sqrt{\frac{2n}{\log 2}}\right)}{C^{\frac{1}{4}} n^{\frac{3}{4}} (\log 2)^n}$

SetSeqSetL := [T, {T = Set(S), S = Sequence(U, card ≥ 1), U = Set(Z, card ≥ 1)}, labeled];

Hierarchies

A000041 = 1, 2, 3, 5, 7, 11, 15, ...

Partition numbers.

Euler transform of A000012 = 1, 1, 1, ...

G.f.: $\prod_{k=1}^{\inf} \frac{1}{1-x^k}$.

$a(n) = \frac{1}{n}\sum_{k=1}^{n-1} c(k)\ A000012(n-k)$, with $c(n) = \sum_{k|n} k\ A000012(k)$.

$a(n) = \sum_{i=1}^{P(n)} 1$.

$a(n) \asymp \frac{1}{4n\sqrt{3}} e^{(\pi\sqrt{(2n/3)})}$.

SetSet := [T, {T = Set(U, card ≥ 1), U = Set(Z, card ≥ 1)}, unlabeled];

A005651 = 1, 3, 10, 47, 246, 1602, ...

Sum of multinomial coefficients.

E.g.f.: $\prod_{k=1}^{\inf} \frac{1}{1-\frac{x^k}{k!}}$.

$a(n) = \sum_{k=1}^{n} C(n-1, k-1)\ c(k)\ a(n-k)$, where $c(n) = \sum_{k|n} (n-1)!\ k\ k!^{-n/k}$.

$a(n) = \sum_{i=1}^{P(n)} \frac{n!}{\prod_{j=1}^{T(i,n)} t(i,j)!}$.

Dissected Hierarchies

A001970 = 1, 3, 6, 14, 27, 58, 111, ...

A143463 = 1, 4, 20, 133, 1047, 9754, ...

Partitions of partitions.	Number of dissected hierarchies that can be formed from n labeled elements.
Euler transform of A000041.	Exp transform of A005651.
G.f.: $\prod_{k=1}^{\inf} \frac{1}{(1-x^k)^{A000041(k)}}$	E.g.f.: $\frac{1}{\exp(x)} \exp\left(\prod_{k=1}^{\inf} \frac{1}{1-\frac{x^k}{k!}}\right)$.
	$a(n) = \sum_{i=1}^{B(n)} \prod_{j=1}^{Z(i)} A005651(\Vert U(j)\Vert)$.
$a(n) = \frac{1}{n} \sum_{k=1}^{n-1} c(k)\ A000041(n-k)$, with $c(n) = \sum_{k\mid n} k\ A000041(k)$.	$a(n) = \sum_{k=1}^{n} C(n-1,k-1)\ A005651(k)\ a(n-k)$.
	$a(n) \asymp C_1 (\frac{1}{n})^{(\frac{3}{4})} \exp\left(\frac{-1}{(\frac{1}{n})^{(\frac{1}{2})}}\right)^{C_2}$ $C1 \approx 0.003513007407$, $C2 \approx -3.180866173225$.
SetSetSetU := [T, {T = Set(S), S = Set(U, card $\geq$ 1), U = Set(Z, card $\geq$ 1)}, unlabeled];	

Table 2: Rankings and hierarchies for elements.

5.1 Partitioning

Rankings and hierarchies obviously follow from a partitioning of a set of elements. The subsets are distributed over the levels. For several structures mentioned here we are able to give an explicit formula on the basis of this partitioning, see table 2. In such a formula we always have a sum $\sum_{i=1}^{P(n)}$ over some expression. This sum indicates to run over all $P(n)$ possible integer partitions of n. The i-th partition is written as $[t(i,1), t(i,2), \ldots, t(i,j), \ldots, t(i,T(i,n))]$ where $T(i,n)$ is the number of parts of the i-th partition. Now, certain parts may be equal $t(i,j) = t(i,j') = t(i,j'') = \ldots$. We write $t(i,j) = d(i,s)$ and say that the different part $d(i,s)$ appears with multiplicity $m(i,s)$. Then an equivalent representation of the i-th partition is $[m(i,1) \times d(i,1), m(i,2) \times d(i,2), \ldots, m(i,s) \times d(i,s), \ldots, m(i,D(i,n)) \times d(i,D(i,n))]$ where $D(i,n)$ is the number of different parts of the i-th partition.

Such a partition-based formula can be found for several other important combinatorial structures and numbers, see for example A000012 (natural numbers), A000045 (Fibonnacci numbers), A000110 (Bell numbers), A007318 (binomial coefficients), A008277 (Stirling numbers of the second kind).

5.2 A034691: Explicit counting formula

Our following derivation of a counting formula shall serve as representative for the full set of derivations which is not included into this paper. In [9] no explicit counting formula for the unlabeled case of A000079 was provided. A000079 is a dissected hierarchy for n unlabeled elements. Thus, we distribute the n unlabeled elements over up to n subhierarchies. In order to do so, we have to carry out an integer partitioning of n and if we want to take into account all possibilities then we have a loop over all partitions. This will be the first loop. Let us pick a certain partition i which has the form $[t(i,1), t(i,2), \ldots, t(i,j), \ldots, t(i,n)]$ with $t(i,1) + t(i,2) + \ldots + t(i,n) = n$. E.g. $n = 5$ and $2+3 = 5$. For example, $t(i,2)$ is the number of elements in the second subhierarchy. Then we have to distribute the $t(i,2)$ elements over up to $t(i,2)$ possible levels. E.g. for $t(i,2) = 3$ we have four subhierarchies: 3; 2|1; 1|2; 1|1|1.

We call this distributing over levels the second partition stage. How many possibilities for the distribution are there? We put $t(i,j)$ unlabeled elements into l boxes with $1 \leq l \leq t(i,j)$. There are $C(t(i,j), l)$ possibilities for doing this and since we have $l = 1, ..., t(i,j)$ levels we have to sum over l which results in the simple factor $2^{(t(i,j)-1)}$.

So far, it was not so complicated. But we consider two multiple hierarchies as identical if they have the same subhierarchies. The problem comes from parts which arise repeatedly in the first partition stage. E.g. for $n = 5$ we can have the partition $[1, 2, 2]$, that is two times the part "2". Each "2" can be partitioned, i.e. distributed over one or over two levels according to the partitions [2] and $[1, 1]$. The cartesian product among [2] and $[1, 1]$ counts the totality of all possibilities and is [2],[2]; [2],[1|1]; [1|1],[2]; [1|1],[1|1]. We see that the case [2],[1|1] = [1|1],[2] occurs two times, but we want to count this case only one time.

Now we go over from parts $t(i,j)$ to the different parts $d(i,s)$ and their multiplicities $m(i,s)$. What we are counting here is the number of different combinations with repetitions of $d(i,j)$ elements in the order $m(i,j)$. There are $C(d(i,j) + m(i,j) - 1, m(i,j))$ possibilities to do that. Finally we have for $A034691(n)$

$$A034691(n) = \sum_{i=1}^{P(n)} \prod_{s=1}^{D(i,n)} C(2^{(d(i,s)-1)} + m(i,s) - 1, m(i,s)). \tag{1}$$

5.3 Transforms among rankings/hierarchies

A transform $\mathbf{T}$ of an integer sequence $a(n)$ maps $a(n)$ onto another integer sequence $b(n)$ [2]. Now, if $a(n)$ counts a single structure and if we apply a

Table 3: Transformations among sequences which arise from rankings and hierarchies among elements.

	unlabeled	*labeled*
Ranking	A000079 = Invert(A000012)	A000670 = Stirling2(A000142)
Dissected ranking	A034691 = Euler(A000079)	A075729 = Exp(A000670)
Hierarchy	A000041 = Euler(A000012)	A005651 = unknown
Dissected hierarchy	A001970 = Euler(A000041)	A143463 = Exp(A005651)

suitable transform **T**, then we will get a transformed sequence $b(n)$ which then counts the corresponding dissected structure. If we work on a set $\mathcal{U}$ of unlabeled elements, then we have to apply the Euler transform. For a set $\mathcal{L}$ of labeled elements we use the exponential transform. Therefore, we gain the pairwise transform relations like "A034691 is the Euler transform of A000079", see table 3. In addition, in tables 3 and 6 we recognize further transformation relations among our sequences.

The first questions is for the transformation relation among a ranking and its corresponding hierarchy. Obviously, from each ranking we can clone a hierarchy if we just request a monotonously decreasing occupation of ranks with the rank number. Unfortunately, this condition is not reflected in the transforms presently known. Thus, it seems that there is know transform relation among e.g. the ranking A000079 and its corresponding hierarchy A000041.

A second interesting question targets at the transform relation among a certain structure build from unlabeled elements on the one hand and the same structure build from labeled elements on the other hand. If we do not expect a single transform but allow for a sequence of transforms then we eventually can find a path from the unlabeled world to the labeled one or vice versa. [1]

The third question regards the relation between a structure formed from elements and the same structure build from sets of elements. Again we have to discern whether the elements are labeled or unlabeled. For the labeled case we observe three times the Stirling2 transform. For example, we have that A083355 is the Stirling2 transform of A000670 which follows from the definition A000670 of A08335. Perturbingly, A109186 seems not to be connected to A075729 via the Stirling2 transform. For the unlabeled case, we have no findings.

[1] For example, the inverse invert transform, see section 8, of A000079 results in the basic sequence $1, 1, 1, 1, 1, 1, \ldots$ and its first Eulerian transform gives A000142, that is the factorial numbers. From them we arrive at A000670 via the Stirling2 transform.

6 Rankings and Hierarchies for Sets

Table 4 gives examples for rankings and hierarchies build from sets. The sets are formed either from unlabeled or labeled elements. In total eight examples are provided according to the four types of rankings and the four types of hierarchies as introduced in section 3.

Table 5 assembles the counting formulas for the structures from table 4. We do not give derivations for all these formulas. The authorships of the formulas can be looked up in the OEIS under the corresponding OEIS numbers which are all given here.

Table 4: Examples for rankings and hierarchies among sets of elements.

	unlabeled	labeled
Ranking	A055887: {1,1,1}; {1,1}{1}; {1}:{1,1}; {1,1}:{1}; {1}{1}{1}; {1}{1}:{1}; {1}:{1}{1}; {1}:{1}:{1}	A083355: {1,2,3}; {1,2},{3}; {1,2}:{3}; {3}:{1,2}; {1,3},{2}; {1,3}:{2}; {2}:{1,3}; {2,3},{1}; {2,3}:{1}; {1}:{2,3}; {1},{2},{3}; {1}:{2}:{3}; {3}:{1}:{2}; {2}:{3}:{1}; {1}:{3}:{2}; {2}:{1}:{3}; {3}:{2}:{1}; {1},{2}:{3}; {1},{3}:{2}; {2},{3}:{1}; {1}:{2},{3}; {2}:{1},{3}; {3}:{1},{2}
Dissected ranking	A104525: {1},{1},{1}; {1}|{1}|{1}; {1,1,1}; {1}:{1}:{1}; {1}|{1}:{1}; {1,1}:{1}; {1}:{1},{1}; {1}|{1},{1}; {1},{1}:{1}; {1}:{1,1}; {1}|{1,1}; {1},{1,1}	A109186: {1,2,3}; {1,3},{2}; {1}:{2}:{3}; {1}:{2,3}; {1,2}:{3}; {1}|{2}:{3}; {2,3}|{1}; {1},{2}|{3}; {2},{3}:{1}; {3}:{1},{2}; {2,3},{1}; {2}:{1}:{3}; {2}:{1,3}; {1,3}:{2}; {1}|{3}:{2}; {1,2}|{3}; {2},{3}|{1}; {1},{3}:{2}; {2}:{1},{3}; {1}|{2}|{3}; {1,2},{3}; {1}:{3}:{2}; {3}:{1,2}; {2,3}:{1}; {2}|{3}:{1}; {1,3}|{2}; {1},{3}|{2}; {1},{2}:{3}; {1}:{2},{3}; {1},{2},{3}; {3}:{2}:{1}; {1}:{3}|{2}; {2}:{3}:{1}; {3}|{2}:{1}; {3}:{1}:{2}; {3}|{1}:{2}
Hierarchy	A141199: {1,1,1}; {1,1}{1}; {1}:{1,1}; {1,1}:{1}; {1}{1}{1}; {1}{1}:{1}; {1}:{1}:{1}	A140585: {1,2,3}; {1,2},{3}; {1},{2},{3}; {1,3},{2}; {2,3},{1}; {1,2}:{3}; {1,3}:{2}; {2,3}:{1}; {3}:{1,2}; {2}:{1,3}; {1}:{2,3}; {1},{2}:{3}; {3},{1}:{2}; {2},{3}:{1}; {1}:{2}:{3}; {3}:{1}:{2}; {2}:{3}:{1}; {1}:{3}:{2}; {2}:{1}:{3}; {3}:{2}:{1}
Dissected hierarchy	A144791: {1},{1},{1}; {1}|{1}|{1}; {1,1,1}; {1}:{1}:{1}; {1}|{1}:{1}; {1,1}:{1}; {1}|{1,1}; {1},{1}:{1}; {1}:{1,1}; {1}|{1,1}; {1},{1,1}	A144792: {1,2,3}; {1,3},{2}; {1}:{2}:{3}; {1}:{2,3}; {1,2}:{3}; {1}|{2}:{3}; {2,3}|{1}; {1},{2}|{3}; {2},{3}:{1}; {2,3},{1}; {2}:{1}:{3}; {2}:{1,3}; {1,3}:{2}; {1}|{3}:{2}; {1,2}|{3}; {2},{3}|{1}; {1},{3}:{2}; {1}|{2}|{3}; {1,2},{3}; {1}:{3}:{2}; {3}:{1,2}; {2,3}:{1}; {2}|{3}:{1}; {1,3}|{2}; {1},{3}|{2}; {1},{2}:{3}; {1},{2},{3}; {3}:{2}:{1}; {1}:{3}|{2}; {2}:{3}:{1}; {3}|{2}:{1}; {3}:{1}:{2}; {3}|{1}:{2}

Sets	
unlabeled	**labeled**

Rankings	
A055887 = 1, 3, 8, 22, 59, 160, 431, ...	A083355 = 1, 4, 23, 175, 1662, 18937, ...
Number of ordered partitions of partitions.	Number of preferential arrangements for the set partitions of the n-set [1,2,3,...,n].
Invert transform of the partitions numbers A000041.	Stirling transform of the preferential arrangements A000670.
G.f.: $\frac{1}{2-\prod_{k=1}^{\inf}\frac{1}{1-x^k}}$.	E.g.f.: $\frac{1}{2-\exp(\exp(x)-1)}$.
	$a(n)=\sum_{i=1}^{n}\sum_{j=1}^{i} j!\ S_2(i,j)\ S_2(n,i)$.
$a(n)=\sum_{k=1}^{n} a(n-k)\ A000041(k)$.	
	$a(n)=\frac{\exp(-n)^{\ln(\ln(1+\ln(2)))}}{2(1+\ln(2))\ln(1+\ln(2))}$.
SeqSetSetU := [T, {T = Sequence(S), S = Set(U, card ≥ 1), U=Set(Z, card ≥ 1), unlabeled];}	SeqSetSetL := [T, {T = Sequence(S), S = Set(U, card ≥ 1), U = Set(Z, card ≥ 1)}, labeled];

Dissected Rankings	
A104525 = 1, 4, 12, 40, 123, 395, 1227, ...	A109186 = 1, 5, 36, 340, 3968, 55045, ...
Number of hierarchical orderings among the parts of the integer partitions of the integer n.	Number of hierarchical orderings among the subsets of the set partitions of the n-set.
Euler transform of A055887.	Exp transform of A083355, Stirling transform of A075729.
E.g.f.: $\prod_{n=1}^{\inf}\frac{1}{(1-x^n)^{A055887(n)}}$.	E.g.f.: $\exp\left(-\frac{\exp(\exp(x)-1)-1}{\exp(\exp(x)-1)-2}\right)$.
$a(n)=\frac{1}{n}\sum_{k=1}^{n-1} c(k)\ A055887(n-k)$, with $c(n)=\sum_{k\mid n} k\ A055887(k)$.	$a(n)=\sum_{k=1}^{n} S_2(n,k)\ A075729(n)$.

	$a(n) = \sum_{k=1}^{n} C(n-1,k-1)\ a(n-k)\ A083355(k)$.
SetSeqSetSetU := [V,V = Set(T), T = Sequence(S, card $\geq$ 1), S = Set(U, card $\geq$ 1), U = Set(Z, card $\geq$ 1), unlabeled];	SetSeqSetSetL := [V,V = Set(T), T = Sequence(S, card $\geq$ 1), S = Set(U, card $\geq$ 1), U = Set(Z, card $\geq$ 1), labeled];

Hierarchies

A141199 = 1, 3, 7, 17, 38, 87, 191,...	A140585 = 1, 4, 20, 129, 1012, 9341, ...
The number of hierarchically ordered partitions of partitions.	Total number of all hierarchical orderings for all set partitions of n.
	Stirling transform of A005651.
G.f.: $\frac{1}{\prod_{i=1}^{\inf}\left(1-\frac{x^i}{\prod_{j=1}^{i}(1-x^j)}\right)}$.	E.g.f.: $\frac{1}{\prod_{k=1}^{\inf}\left(1-\frac{(exp(x)-1)^k}{k!}\right)}$.
	$a(n) = \sum_{k=1}^{n} S_2(n,k)\ A005651(n)$.
$a(n) \asymp 3.788562346\ \ (\exp(-n))^{(-\ln(2))}$.	

Dissected Hierarchies

A144791 = 1, 4, 11, 34, 93, 269, 735, ...	A144792 = 1, 5, 33, 282, 2938, 36029, ...
Euler transform of A141199.	Exp transform of A140585, Stirling transform of A143463.
G.f.: $\frac{1}{\prod_{k=1}^{\inf}(1-x^k)^{A141199(k)}}$.	E.g.f: $\frac{1}{\exp(1)}\ \exp\left(\frac{1}{\prod_{k=1}^{\inf}\left(1-\frac{(\exp(x)-1)^k}{k!}\right)}\right)$.
$a(n) = \frac{1}{n}\sum_{k=1}^{n-1} c(k)\ A141199(n-k)$, with $c(n) = \sum_{k\mid n} k\ A141199(k)$.	$a(n) = \sum_{k=1}^{n} C(n-1,k-1)\ a(n-k)\ A140585(k)$.
	$a(n) = \sum_{k=1}^{n} S_2(n,k)\ A143463(n)$.

Table 5: Rankings and hierarchies for sets.

6.1 A104525: Euler transform

In Theorem 2 of [9] it is shown that A034691 is the Euler transform of 2^{n-1}. The argument can be applied in the same way to A104525 which is the Euler transform of A055887, the number of ordered partitions of partitions. To obtain the ordered partition of partitions of n unlabeled elements we proceed as follows. For a given integer partition of n we distribute n unlabeled elements $1, 1, 1, \ldots$ into subsets as prescribed by the partition. E.g. for $n = 5$ and its partition $[1, 2, 2]$ we get $\{1\}, \{1, 1\}, \{1, 1\}$. Subsequently we form all permutations of the subsets which are for our example $\{1\} : \{1, 1\} : \{1, 1\}$; $\{1, 1\} : \{1, 1\} : \{1\}$; $\{1, 1\} : \{1\} : \{1, 1\}$. Summing up the numbers of all permutations for all partitions of n gives the desired number of ordered partitions of partitions. To obtain their Euler transform we carry out a second partition after the first. We split up the subsets into groups. Within each group we establish again all possible permutations. For our example we gain {1}: {1,1}: {1,1}; {1,1}: {1,1}: {1}; {1,1}: {1}: {1,1} and {1}: {1,1}|{1,1}; {1,1}: {1}|{1,1}; {1,1}: {1,1}|{1}; {1,1}|{1,1}|{1}. This process has to be applied to all integer partitions of n in order to get A104525. This example has been given explicitly to demonstrate the principle of the Euler transform, that is the second partition. This principle is reflected in the form of the generating function g.f. of the transformed sequence $b(n)$, see equation (5) in section 8. From that equation we immediately get the g.f. of A104525 as stated in table 5. In addition we get a recursive expression for $b(n)$, because of equation (4).

6.2 A109186: Exponential transform

For our labeled structures the Exp(onential) transform is the counterpart to the Euler transform in the unlabeled case. It also corresponds to a second partition whereby the subsequent permutation acts on labeled elements. As before for the Euler transform, the exponential generating function e.g.f. of the Exp transform $b(n)$ is known, its formula is (7). Furthermore, a recursive formula for $b(n)$ is at hand, that is (6). From these two formulas we have at once for A109186 the results shown in table 5.

7 Discussion

Several tasks follow from the present state. Some explicit or implicit counting formulas are still missing and should be supplied. New structures can be envisaged like strict hierarchies where the occupation number is strictly decreasing with increasing rank number. Also of interest are structures where

Table 6: Transformations among sequences which arise from rankings and hierarchies among sets of elements.

	unlabeled	*labeled*
Ranking	A055887 = Invert(A000041)	A083355 = Stirling2(A000670)
Dissected ranking	A104525 = Euler(A055887)	A109186 = Exp(A083355)
Hierarchy	A141199 = unknown	A140585 = Stirling2(A005651)
Dissected hierarchy	A144791 = Euler(141199)	A144792 = Exp(140585)
		A144792 = Stirling2(A143463)

empty levels are allowed. [2]

Once we have theoretical descriptions of structures, then we are interested in their realizations in nature and social reality. Those realizations should be identified and studied in terms of enumerative combinatorics, probability theory and social network analysis. Many quantitative features of our rankings and hierarchies are still open for characterization. For example, what is the average rank of an arbitrary element? What is the average height and width of a structure? For A034691 and A075729 a few of these question were tackled in [9].

These are static properties. We are heading for dynamic properties also. How do rankings and hierarchies evolve? A first approach would be to analyze and count the possible paths from one structure to another. By path we mean the necessary movements of elements among the levels. A first step into that direction was done by counting one-step transitions for A034691 [3]. Concerning such transitions among partitions one may look up A093694, A093695, A094533, A096541, A094251, A096586, A096586, A000070. Quantitative sociology has developed its notions and formulas for analyzing social networks [11]. Social networks are described as directed graphs in the graph theoretical sense. Characteristics like nodal degree, density of a graph and centrality are

[2] Sequences A000262 and A001700 of the OEIS belong to this category: "Consider the distribution of n unlabeled elements "1" onto n levels where empty levels are allowed. In addition, the empty levels are labeled. Their names are 0_1, 0_2, 0_3, etc. This sequence gives the total number of such distributions. If the empty levels are unlabeled ("0"), then the answer is A001700 Let the colon ":" separate two levels. Then, for example, for $n = 3$ we have $a(3) = 13$ arrangements, namely 111 : 0_1 : 0_2, 0_1 : 111 : 0_2, 0_1 : 0_2 : 111, 111 : 0_2 : 0_1, 0_2 : 111 : 0_1, 0_2 : 0_1 : 111, 11 : 1 : 0, 11 : 0 : 1, 0 : 11 : 1, 1 : 11 : 0, 1 : 0 : 11, 0 : 1 : 11, 1 : 1 : 1."

[3] "For given n (= number of elements) we consider two hierarchies H1 and H2. We ask whether a one-step transition is possible from H1 to H2 (if it is possible, then there is also a one-step transition from H2 to H1). In a one-step transition just one single element is moved from its position in H1 to its position in H2."

at our disposal. A ranking or a hierarchy can be described in graph theoretical terms, too. One even can imagine to introduce directed connections among the elements.

The author is well aware that what is called a hierarchy here is not what is called a hierarchy in sociology. [4] However, the author's believe is that questions for the growth and decay of hierarchies can be treated by quantitative methods to a certain extend. In particular, the question for optimal structures should be accessible by model theoretic methods. Dissected hierarchies are a prominent example: Individuals may tend to separate into dissected groups in order to minimize hierarchical structures. On the other hand, a hierarchical organization of a group may stabilize the group and may strengthen its capabilities. Then the question arises: What is the optimal subgroup size and the optimal hierarchy height for given n? Mathematical and computer-based sociology will provide us with further insights into sociological structure dynamics. For the enumeration of coalitions, see [12].

Acknowledgements Many thanks to Neil Sloane, AT&T Shannon Lab, USA, for his patience and efforts with my contributions to the OEIS. Many thanks to Fred Lunnon, National University of Ireland, Maynooth, for interesting discussions and valuable critic of the manuscript. Also many thanks to Sui Sun Cheng, Tsing Hua University, Taiwan, for his friendly encouragement.

8 Appendix

The invert transform $b(n)$ of a sequence $a(n)$ is defined in terms of generating functions. If $f(x)$ is the generating function of $a(n)$ and $g(x)$ is the generating function of $b(n)$ then

$$1 + \sum_{n=0}^{\infty} b(n)\ x^n = \frac{1}{1 - \sum_{n=0}^{\infty} a(n)\ x^n}. \tag{2}$$

Let the sequence $c(n)$ be given by

$$c(n) = \sum_{k|n} k\ a(k) \tag{3}$$

where the sum is taken for all k which divide n. Then the Euler transform $b(n)$ of a sequence $a(n)$ is defined as

$$b(n) = \frac{1}{n} \sum_{k=1}^{n-1} c(k) a(n-k) \tag{4}$$

[4]For a recent discussion of the notion "hierarchy" in natural sciences and sociology see [5].

where $b(1) = c(1)$. An equivalent definition of the Euler transform is

$$1 + \sum_{n=1}^{\inf} b(n)\ x^n = \prod_{k=1}^{\inf} \frac{1}{(1-x^i)^{a(i)}}. \tag{5}$$

A further transform is given by the exp(onential) transform. If $b(n)$ denotes the exp transform of $a(n)$, then

$$b(n) = \sum_{k=1}^{n} C(n-1, k-1) a(n) b(n-k). \tag{6}$$

In terms of generating functions the exp transform of a sequence $a(n)$ is defined as

$$1 + \sum_{n=1}^{\inf} \frac{b(n)\ x^n}{n!} = \exp\left(\sum_{n=1}^{\inf} \frac{a(n)\ x^n}{n!}\right). \tag{7}$$

The Stirling transform $b(n)$ of a sequence $a(n)$ is defined as

$$b(n) = \sum_{k=1}^{n} S_2(n,k) a(k). \tag{8}$$

As for the invert transform, one may define the Stirling transform via generating functions. Then, if $f(x)$ is the exponential generating function of $a(n)$ and $g(x)$ is the exponential generating function of $b(n)$ then

$$g(x) = f(\exp(x) - 1). \tag{9}$$

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Received: April, 2009

Table 7: List of symbols and abbreviations.

$\asymp$	asymptotic to.
$\|\mathcal{U}\|$	number of elements of set $\mathcal{U}$.
$\times$	$x \times y = xy$.
$\sum_{i=1}^{P(n)}$	sum over the $P(n)$ integer partitions of n.
$\mathbf{T}$	a transform $b(n) = \mathbf{T}(a(n))$ of integer sequence $a(n)$.
$\mathcal{A}$	a list of n levels $[l_1, l_2, l_3, \ldots, l_n]$.
$\mathcal{U}$	a set of n unlabeled elements $\{1, 1, 1, \ldots\}$.
$\mathcal{L}$	a set of n labeled elements $\{1, 2, 3, \ldots, n\}$.
$a(n)$	(a counting formula for) an integer sequence.
$b(n)$	another integer sequence, transform of $a(n)$.
$d(i, s)$	s-th different part of the i-th partition of n.
$m(i, s)$	multiplicity of the s-th different part $d(i, s)$
in the i-th integer partition of n.	
$t(i, j)$	j-th part of the i-th partition of n, $\sum_{j=1}^{T(i,j)} t(i, j) = n$.
$o(l_j)$	occupation number of level l_j.
$B(n)$	Bell numbers.
$C(n, k)$	binomial coefficients.
$D(i, n)$	number of different parts of the i-th partition of n, $\sum_{s=1}^{D(i,n)} m(i, s)d(i, s) = n$.
$P(n)$	number of integer partitions of n.
$S_2(n, k)$	Stirling numbers of the second kind.
$T(i, n)$	number of parts of the i-th partition of n. $\sum_{s=1}^{T(i,n)} t(i, j) = n$.
$\sigma(n)$	the number of divisors of n.
E.g.f	exponential generating function.
G.f.	generating function.
OEIS	On-Line Encyclopedia of Integer Sequences.