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Exponential Stabilizability of Linear Systems with Multiple Delays on States and Controls

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Abstract

Exponential stabilizability of linear control systems with multiple delays on states and controls is studied in this paper. Based on the Lyapunov method, we establish new criteria that ensure the exponential stability of the closed-loop system with memoryless state feedback control or instantaneous feedback control. The criteria are derived in terms of linear matrix inequalities, which allows to compute simultaneously two bounds that characterize the exponential stability rate of the solution. Our results are illustrated with numerical examples.

Keywords: Exponential stabilizability; Multiple delays; Lyapunov functional; Linear matrix inequalities

1 Introduction

The stabilization problem of linear control systems with delays on state and control has been an interesting and challenging problem in recent years (see in [11]). Although many stabilizability criteria for linear systems only with delays on control have been proposed, there are few stabilizability criteria for linear systems with delays on state and control [2, 5, 8]. Moreover, the delayed control part in these existing results are considered as a perturbation rather than an effective control input. So, stabilizability criteria, where the delayed control part is an effective control input, for linear systems with delays on state and control will be interesting results. To the best of our knowledge, there are only two results like this [7, 12]. In this paper, inspired by the result in [10] for exponential stability of linear systems with multiple delays, we proposed a new exponential stabilizability criterion for linear control systems with multiple delays on state and control.

$$\begin{cases} \dot{x}(t) & = A_0x(t) + \sum_{k=1}^{m} A_kx(t-\tau_k) + B_0u(t) + \sum_{k=1}^{m} B_ku(t-\tau_k) \\ x(t) & = \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{1.1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control $A_k, B_k, k = 1, 2, ..., m$ are given matrices, $\tau_k, k = 1, 2, ..., m$ are delays satisfying $0 < \tau_k \leq \tau$ and initial condition is $\phi(\theta) \in C([-\tau, 0], \mathbb{R}^n)$.

By combining the Lyapunov-Krasovski stability theory and a parameterized neutral model transformation, we have obtained a sufficient condition for δ-stabilizability of system (1.1) with a memoryless feedback control and a sufficient condition for δ-stabilizability of system

$$\dot{x}(t) = A_0x(t) + \sum_{k=1}^{m} A_kx(t-\tau_k) + B_0u(t)$$

with an instantaneous feedback control term $u(t) = Kx(t) + \sum_{i=1}^{m} K_ix(t-\tau_i)$.

The following well-known lemmas are needed for main results

Lemma 1.1. *Assume that $S \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Then for every $Q \in \mathbb{R}^{n\times n}$,*

$$2\langle Qy, x\rangle - \langle Sy, y\rangle \leq \langle QS^{-1}Q^Tx, x\rangle, \quad \forall x, y \in \mathbb{R}^n.$$

Lemma 1.2. *For any constant matrix $M \in \mathbb{R}^{m\times n}, M = M^T > 0$, scalar $\sigma > 0$, vector function $w : [0, \sigma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, then*

$$\Big(\int_0^\sigma w(s)ds\Big)^T M \Big(\int_0^\sigma w(s)ds\Big) \leq \sigma \int_0^\sigma w^T(s)Mw(s)ds.$$

2 Main Results

For matrices $X, W, F_{11}, F_{22}, F_{33}, F_{12}, F_{13}, F_{23}, Y_k$, $k = 1, 2, ..., m$, matrix Z and real numbers $\alpha > 1, \delta > 0$, we denote

$$\Sigma_1 = \sum_{k=1}^{m} \Big\{(\delta I + A_0)X + X(\delta I + A_0)^T + B_0Z + Z^TB_0^T + Y_k + Y_k^T + W + \tau_kF_{11})\Big\},$$

$\Phi_k = XA_0^T + Z^TB_0^T + Y_k^T + F_{12}, \quad \Psi_k = mA_kX + mB_kZ - Y_k + \tau_kF_{13},$

$\Gamma_k = A_kX+B_kZ-Y_k+F_{23}, \Lambda_k = -{\tau_k}^{-1}(\alpha X - F_{22}), M_k = (\alpha\tau_kY_k, 0, ..., 0)^T_{2m+1,1},$

$\Omega_{m+2,m+2} = -e^{-2\delta\tau_1}W + \tau_1F_{33}, ..., \Omega_{2m+1,2m+1} = -e^{-2\delta\tau_m}W + \tau_1F_{33}.$

$$\Omega = \begin{pmatrix} \Sigma_1 & \Phi_1 & \Phi_2 & ... & \Phi_m & \Psi_1 & \Psi_2 & ... & \Psi_{m-1} & \Psi_m \\ \star & \Lambda_1 & 0 & ... & 0 & \Gamma_1 & A_2X+B_2Z & ... & A_{m-1}X+B_{m-1}Z & A_mX+B_mZ \\ \star & \star & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \star & \vdots & \star & \ddots & 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \vdots & \cdots & \star & \Lambda_m & A_1X+B_1Z & A_2X+B_2Z & ... & A_{m-1}X+B_{m-1}Z & \Gamma_m \\ \star & \vdots & \vdots & \vdots & \star & \Omega_{m+2,m+2} & 0 & ... & 0 & 0 \\ \star & \vdots & \vdots & \vdots & \vdots & \star & \ddots & \ddots & \vdots & \vdots \\ \star & \vdots & \vdots & \vdots & \vdots & \vdots & \star & \ddots & \ddots & \vdots \\ \star & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \star & \ddots & \vdots \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \Omega_{2m+1,2m+1} \end{pmatrix}, \tag{2.1}$$

We have the following theorem.

Theorem 2.1. *For given $\tau_k > 0, k = 1, 2, ..., m$, system* (1.1) *is δ-stabilizable if there exist positive definite symmetric matrices $X, W, F_{11}, F_{22}, F_{33}$ and any matrices Z, Y_k, $k = 1, 2, ..., m$, F_{12}, F_{13}, F_{23}, a number $\alpha > 1$ such that and the following LMIs hold*

$$\Xi(B_0Z, B_1Z, ..., B_mZ) = \begin{pmatrix} \Omega & M_1 & M_2 & ... & M_m \\ \star & -\alpha\tau_1X & 0 & ... & 0 \\ \star & \star & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \star & \star & ... & \star & -\alpha\tau_mX \end{pmatrix} < 0, \tag{2.2}$$

$$-X + F_{22} < 0, \tag{2.3}$$

$$\overline{F} = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ \star & F_{22} & F_{23} \\ \star & \star & F_{33} \end{pmatrix} > 0, \tag{2.4}$$

The feedback control is $u(t) = ZX^{-1}x(t)$. Moreover,there exists a positive constant N such that the solution $x(t, \phi)$ of the closed-loop system satisfies

$$||x(t, \phi)|| \leq N||\phi||e^{-\delta t}, \quad \forall t \geq 0.$$

Proof. Putting $K = ZX^{-1}$. The system (1.1) with control $u(t) = Kx(t)$ is

$$\dot{x}(t) = (A_0 + B_0K)x(t) + \sum_{k=1}^{m}(A_k + B_kK)x(t-\tau_k) \tag{2.5}$$

Let us denote $G_k = Y_kX^{-1}, \qquad P = X^{-1}, \qquad T = X^{-1}WX^{-1},$

$$\mathcal{D}_k(x_t) = x(t) + e^{-\delta t}\int_{t-\tau_k}^{t} G_kx(s)e^{\delta s+\delta\tau_k}ds, \qquad \overline{P} = diag\{P,P,P\}.$$

Consider the following Lyapunov functional associated with the system (2.5):

$$V^* = V + V_5 \tag{2.6}$$

where $V = \sum_{k=1}^{m} V_{1k} + V_{2k} + V_{3k} + V_{4k}$,
$V_{1k} = \mathcal{D}_k^T(x_t)P\mathcal{D}_k(x_t)$,
$V_{2k} = \alpha e^{-2\delta t}\int_{t-\tau_k}^{t}\int_{s}^{t} e^{2\delta(u+\tau_k)}x^T(u)G_k^TPG_kx(u)duds$,
$V_{3k} = e^{-2\delta t}\int_{t-\tau_k}^{t} e^{2\delta s}x^T(s)Tx(s)ds$,
$V_{4k} = e^{-2\delta t}\int_{0}^{t}\int_{s-\tau_k}^{s}\begin{pmatrix} e^{\delta s}x(s) \\ e^{\delta(u+\tau_k)}G_kx(u) \\ e^{\delta s}x(s-\tau_k)\end{pmatrix}^T \overline{P}.\overline{F}.\overline{P}\begin{pmatrix} e^{\delta s}x(s) \\ e^{\delta(u+\tau_k)}G_kx(u) \\ e^{\delta s}x(s-\tau_k)\end{pmatrix}duds$,
$V_5 = \eta x^T(t)x(t)$, η is a positive number that will be chosen later.

Taking the derivative of V_{1k}, we have

$$\begin{aligned}
\dot{V}_{1k} &= 2\mathcal{D}_k^T(x_t)P\dot{\mathcal{D}}_k(x_t)\\
&= 2\Big\{x(t) + e^{-\delta t}\int_{t-\tau_k}^{t} G_kx(s)e^{\delta s+\delta\tau_k}ds\Big\}^TP\\
&\times\Big\{\dot{x}(t) - \delta e^{-\delta t}\int_{t-\tau_k}^{t} G_kx(s)e^{\delta s+\delta\tau_k}ds + e^{-\delta t}(G_kx(t)e^{\delta t+\delta\tau_k} - G_kx(t-\tau_k)e^{\delta t})\Big\}\\
&= 2\Big\{x(t) + e^{-\delta t}\int_{t-\tau_k}^{t} G_kx(s)e^{\delta s+\delta\tau_k}ds\Big\}^TP\\
&\times\Big\{\dot{x}(t) - \delta e^{-\delta t}\int_{t-\tau_k}^{t} G_kx(s)e^{\delta s+\delta\tau_k}ds + G_ke^{\delta\tau_k}x(t) - G_kx(t-\tau_k)\Big\}.
\end{aligned}$$

This follows that

$$\begin{aligned}
\dot{V}_{1k} + 2\delta V_{1k} &= 2\Big\{x(t) + e^{-\delta t}\int_{t-\tau_k}^{t} G_kx(s)e^{\delta s+\delta\tau_k}ds\Big\}^TP\\
&\times\Big\{\dot{x}(t) + \delta x(t) + G_ke^{\delta\tau_k}x(t) - G_kx(t-\tau_k)\Big\}\\
&= 2\Big\{x(t) + e^{-\delta t}\int_{t-\tau_k}^{t} G_kx(s)e^{\delta(s+\tau_k)}ds\Big\}^TP\\
&\times\Big\{(\delta I + A_0 + B_0K + G_ke^{\delta\tau_k})x(t) + \sum_{j=1}^{m}(A_j + B_jK)x(t-\tau_j) - G_kx(t-\tau_k)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= 2x^T(t)\{P(\delta I + A_0 + B_0K + G_k e^{\delta\tau_k})\}x(t) + 2x^T(t)P\sum_{j=1}^{m}(A_j + B_jK)x(t-\tau_j)\\
&\quad - 2x^T(t)PG_kx(t-\tau_k) + 2e^{-\delta t}\Big(\int_{t-\tau_k}^{t} G_kx(s)e^{\delta(s+\tau_k)}ds\Big)^T P\Big(\delta I + A_0 + B_0K\\
&\quad + G_ke^{\delta\tau_k}\Big)x(t) \quad + 2e^{-\delta t}\Big(\int_{t-\tau_k}^{t} G_kx(s)e^{\delta(s+\tau_k)}ds\Big)^T P\sum_{j=1}^{m}(A_j + B_jK)x(t-\tau_j)\\
&\quad - 2e^{-\delta t}\Big(\int_{t-\tau_k}^{t} G_kx(s)e^{\delta(s+\tau_k)}ds\Big)^T PG_kx(t-\tau_k).
\end{aligned}
\tag{2.7}
$$

Using Lemma 1.2, we have

$$
\begin{aligned}
&\dot{V}_{2k} + 2\delta V_{2k}\\
&\leqslant \alpha\tau_k e^{2\delta\tau_k}x^T(t)G_k^TPG_kx(t) - e^{-2\delta t}\int_{t-\tau_k}^{t} e^{2\delta(s+\tau_k)}x^T(s)G_k^TG_kx(s)ds\\
&\quad - {\tau_k}^{-1}(\alpha-1)e^{-2\delta t}\Big(\int_{t-\tau_k}^{t} e^{2\delta(s+\tau_k)}G_kx(s)ds\Big)^T P\Big(\int_{t-\tau_k}^{t} e^{\delta(s+\tau_k)}G_kx(s)ds\Big).
\end{aligned}
\tag{2.8}
$$

By simple computations, we have

$$
\dot{V}_{3k} + 2\delta V_{3k} = x^T(t)Tx(t) - e^{-2\delta\tau_k}x^T(t-\tau_k)Tx(t-\tau_k).
\tag{2.9}
$$

$$
\begin{aligned}
&\dot{V}_{4k} + 2\delta V_{4k}\\
&= \tau_k x^T(t)PF_{11}Px(t) + 2\tau_k x^T(t)PF_{13}Px(t-\tau_k) + \tau_k x^T(t-\tau_k)PF_{33}Px(t-\tau_k)\\
&\quad + 2x^T(t)PF_{12}P\Big(e^{-\delta t}\int_{t-\tau_k}^{t} e^{\delta(s+\tau_k)}G_kx(s)ds\Big)\\
&\quad + 2x(t-\tau_k)PF_{23}P\Big(e^{-\delta t}\int_{t-\tau_k}^{t} e^{\delta(s+\tau_k)}G_kx(s)ds\Big)\\
&\quad + e^{-2\delta t}\int_{t-\tau_k}^{t} e^{2\delta(s+\tau_k)}x^T(s)dsG_k^TPF_{22}PG_kx(s)ds.
\end{aligned}
\tag{2.10}
$$

From (2.7) - (2.10), we have

$$
\begin{aligned}
\sum_{k=1}^{m}\Big(\dot{V}_{1k} + 2\delta V_{1k}\Big) &= 2x^T(t)\Big\{\sum_{k=1}^{m}P(\delta I + A_0 + B_0K + G_ke^{\delta\tau_k})\Big\}x(t)\\
&\quad + \sum_{k=1}^{m}\{2x^T(t)(mP(A_k + B_kK) - PG_k)x(t-\tau_k)\}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\sum_{k=1}^{m}\Big\{e^{-\delta t}\Big(\int_{t-\tau_k}^{t} G_k x(s)e^{\delta(s+\tau_k)}ds\Big)^T P(\delta I + A_0 + B_0K + G_k e^{\delta\tau_k})x(t)\Big\}\\
&- 2\sum_{k=1}^{m}\Big\{e^{-\delta t}\Big(\int_{t-\tau_k}^{t} G_k x(s)e^{\delta(s+\tau_k)}ds\Big)^T PG_k x(t-\tau_k)\Big\}\\
&+ 2\sum_{k=1}^{m}\Big\{e^{-\delta t}\Big(\int_{t-\tau_k}^{t} G_k x(s)e^{\delta(s+\tau_k)}ds\Big)^T P\sum_{j=1}^{m}(A_j + B_jK)x(t-\tau_j)\Big\}.
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
&\sum_{k=1}^{m}\Big(\dot V_{2k} + 2\delta V_{2k}\Big)\\
&\leqslant \sum_{k=1}^{m}\alpha\tau_k e^{2\delta\tau_k}x^T(t)G_k^T PG_k x(t) - \sum_{k=1}^{m}\Big\{e^{-2\delta t}\int_{t-\tau_k}^{t} e^{2\delta(s+\tau_k)}x^T(s)G_k^T G_k x(s)ds\Big\}\\
&-\sum_{k=1}^{m}\Big\{{\tau_k}^{-1}(\alpha-1)e^{-2\delta t}\Big(\int_{t-\tau_k}^{t} e^{2\delta(s+\tau_k)}G_k x(s)ds\Big)^T P\Big(\int_{t-\tau_k}^{t} e^{\delta(s+\tau_k)}G_k x(s)ds\Big)\Big\}.
\end{aligned}
\tag{2.12}
$$

$$
\sum_{k=1}^{m}\Big(\dot V_{3k} + 2\delta V_{3k}\Big) = mx^T(t)Tx(t) - \sum_{k=1}^{m} e^{-2\delta\tau_k}x^T(t-\tau_k)Tx(t-\tau_k)
\tag{2.13}
$$

$$
\begin{aligned}
\sum_{k=1}^{m}\Big(\dot V_{4k} + 2\delta V_{4k}\Big) &= \sum_{k=1}^{m}\tau_k x^T(t)PF_{11}Px(t) + 2\sum_{k=1}^{m}\{\tau_k x^T(t)PF_{13}Px(t-\tau_k)\}\\
&+ \sum_{k=1}^{m}\{\tau_k x^T(t-\tau_k)PF_{33}Px(t-\tau_k)\}\\
&+ 2\sum_{k=1}^{m}\Big\{x^T(t)PF_{12}P\Big(e^{-\delta t}\int_{t-\tau_k}^{t} e^{\delta(s+\tau_k)}G_k x(s)ds\Big)\Big\}\\
&+ 2\sum_{k=1}^{m}\Big\{x(t-\tau_k)PF_{23}P\Big(e^{-\delta t}\int_{t-\tau_k}^{t} e^{\delta(s+\tau_k)}G_k x(s)ds\Big)\Big\}\\
&+ \sum_{k=1}^{m}\Big\{e^{-2\delta t}\int_{t-\tau_k}^{t} e^{2\delta(s+\tau_k)}x^T(s)G_k^T PF_{22}PG_k x(s)ds\Big\}.
\end{aligned}
\tag{2.14}
$$

These follow that

$$
\dot V + 2\delta V \leq \eta(t)^T\Lambda\eta(t) + \sum_{k=1}^{m} e^{-2\delta t}\int_{t-\tau_k}^{t} e^{2\delta(s+\tau_k)}x^T(s)G^T(-P + PF_{22}P)Gx(s)ds.
\tag{2.15}
$$

where $\eta(t) = \begin{pmatrix} x(t) \\ e^{-\delta t}\int_{t-\tau_1}^{t} G_1 x(s) e^{\delta(s+\tau_1)} ds \\ \vdots \\ e^{-\delta t}\int_{t-\tau_m}^{t} G_m x(s) e^{\delta(s+\tau_m)} ds \\ x(t-\tau_1) \\ \vdots \\ x(t-\tau_m) \end{pmatrix}$

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & ... & \Lambda_{1,m+1} & \Lambda_{1,m+2} & \Lambda_{1,m+3} & ... & \Lambda_{1,2m+1} \\ \star & \Lambda_{22} & 0 & ... & 0 & \Lambda_{2,m+2} & P(A_2+B_2K) & ... & P(A_m+B_mK) \\ \star & \star & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \vdots & \star & \ddots & 0 & \vdots & \vdots & \ddots & \vdots \\ \star & \vdots & \vdots & \star & \Lambda_{m+1,m+1} & P(A_1+B_1K) & P(A_2+B_2K) & ... & \Lambda_{m+1,2m+1} \\ \star & \vdots & \vdots & \vdots & \star & \Lambda_{m+2,m+2} & 0 & ... & 0 \\ \star & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \star & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \star & \star & \star & \star & \star & \Lambda_{2m+1,2m+1} \end{pmatrix},$$

$\Lambda_{11}=\sum_{k=1}^{m}\{P(\delta I+A_0+B_0K+G_k)+(\delta I+A_0+B_0K+G_k)^TP+\alpha\tau_kG_k^TPG_k+T+\tau_kPF_{11}P\}$,
$\Lambda_{12}=(\delta I+A_0+B_0K+G_1e^{\delta\tau_1})^TP+PF_{12}P,\ldots,\ \Lambda_{1,m+1}=(\delta I+A_0+B_0K+G_me^{\delta\tau_m})^TP+PF_{12}P$,
$\Lambda_{1,m+2}=mP(A_1+B_1K)-PG_1+\tau_1PF_{13}P,\ \ldots,\ \Lambda_{1,2m+1}=mP(A_m+B_mK)-PG_m+\tau_mPF_{13}P$,
$\Lambda_{22}=-{\tau_1}^{-1}(\alpha-1)P,\ \ldots,\ \Lambda_{m+1,m+1}=-{\tau_m}^{-1}(\alpha-1)P$,
$\Lambda_{1+k,m+1+k}=P(A_k+B_kK)-PG_k+PF_{23}P, \Lambda_{m+1+k,m+1+k}=-e^{-2\delta\tau_k}T+\tau_kPF_{33}P, k=1,2,...,m.$

The inequality (2.3) is derived from $-P + PF_{22}P < 0$. Hence, by Lemma 1.2, we have

$$\dot{V} + 2\delta V \leq \eta(t)^T(\Lambda + \Delta)\eta(t) \tag{2.16}$$

where $\Delta = \begin{pmatrix} 0 & 0 & ... & ... & ... & 0 & ... & 0 \\ 0 & {\tau_1}^{-1}(-P+PF_{22}P) & 0 & ... & ... & 0 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & ... & 0 & {\tau_m}^{-1}(-P+PF_{22}P) & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 \end{pmatrix}$.

Multiplying both sides of the inequality (2.2) by $diag\{X^{-1}, X^{-1}, ..., X^{-1}\}$ and using Schur complement, we have $\Lambda + \Delta < 0$. This implies that there is a positive number $\lambda > 0$ such that

$$\dot{V} + 2\delta V \leq -\lambda\Big(||x(t)||^2 + \sum_{k=1}^{m} \|x(t-\tau_k)||^2\Big). \tag{2.17}$$

Combining with Lemma 1.1, we have

$$\begin{aligned}\dot{V}^* + 2\delta V^* &\le -\lambda\Big(||x(t)||^2 + \sum_{k=1}^{m} \|x(t-\tau_k)||^2\Big) + 2\eta \dot{x}^T(t)x(t) + 2\eta\delta||x(t)||^2 \\ &\le \Big(-\lambda + \eta(2\delta + 2||A_0 + B_0K|| + \sum_{k=1}^{m} ||A_k + B_kK||^2)\Big)||x(t)||^2 \\ &\quad + \sum_{k=1}^{m}(-\lambda + \eta)||x(t-\tau_k)||^2. \end{aligned} \tag{2.18}$$

Choosing $\eta = \frac{\lambda}{2}\min\{\frac{1}{2\delta + 2||A_0+B_0K|| + \sum_{k=1}^{m}||A_k+B_kK||^2)}, 1\}$, we obtain

$$\dot{V}^* + 2\delta V^* \le 0,$$

which implies that $V^* \le V^*(0)e^{-2\delta t}$. Since $\eta||x(t)||^2 \le V^*$, we get

$$||x(t)|| \le \sqrt{\frac{V^*(0)}{\eta}}e^{-\delta t}. \tag{2.19}$$

On the other hand, we have

$$\begin{aligned} V_{1k}(0) &\le ||P||\Big(||x(0)|| + e^{\delta\tau_k}||G_k||||\phi||\int_{-\tau_k}^{0} e^{\delta s}ds\Big)^2 \\ &\le ||X^{-1}||\Big(1 + e^{\delta\tau_k}||Y_kX^{-1}||\frac{1-e^{-\delta\tau_k}}{\delta}\Big)^2||\phi||^2, \end{aligned}$$

$$\begin{aligned} V_{2k}(0) &= \alpha e^{2\delta\tau_k}\int_{-\tau_k}^{0}\int_{s}^{0} e^{2\delta u}x^T(u)(X^{-1})^TY_k^TX^{-1}Y_kX^{-1}x(u)duds \\ &\le \alpha e^{2\delta\tau_k}||Y_k||^2||X^{-1}||^3||\phi||^2\int_{-\tau_k}^{0}\int_{s}^{0} e^{2\delta u}duds \le \alpha e^{2\delta\tau_k}||Y_k||^2||X^{-1}||^3{\tau_k}^2||\phi||^2, \end{aligned}$$

$$V_{3k}(0) \le \|\phi\|^2\|T\|\int_{-\tau_k}^{0} e^{2\delta s}ds \le \frac{1}{2\delta}(1-e^{-2\delta\tau_k})\|X^{-1}\|^2||W||||\phi\|^2, \quad V_4(0) = 0.$$

Let us denote $N_{1k} = ||X^{-1}||\Big(1 + e^{\delta\tau_k}||Y_kX^{-1}||\frac{1-e^{-\delta\tau_k}}{\delta}\Big)^2$,
$N_{2k} = \alpha e^{2\delta\tau_k}||Y_k||^2||X^{-1}||^3{\tau_k}^2$, $N_{3k} = \frac{1}{2\delta}(1-e^{-2\delta\tau_k})\|X^{-1}\|^2||W||$,
and

$$N = \sqrt{\frac{\eta + \sum_{k=1}^{n}(N_{1k} + N_{2k} + N_{3k})}{\eta}}. \tag{2.20}$$

Then, we get

$$\sqrt{\frac{V^*(0)}{\eta}} = N||\phi||,$$

which implies that $||x(t)|| \leq Ne^{-\delta t}||\phi||$. This means the system (1.1) is δ-stabilizable. The proof is completed. □

Remark 1: In case $k = 1$, we will obtain the result in [7]. Moreover, our condition does not require the controllability of pairs (A_0, B_0) or (A_0+A_1, B_0). Hence, the first open problem of Richard in [11] has been solved.

Example 2.2. *Consider system* (1.1) *with* $m = 1$, *and*

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ -0.5 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -2 & -0.5 & 0 & 0 \\ -0.2 & -1 & 0 & 0 \\ 0.5 & 0 & -2 & -0.5 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$B_0 = [1 \quad 1 \quad 1 \quad 0]^T, \quad B_1 = [0 \quad 1 \quad 1 \quad 1]^T.$$

Since $rank[B_0 \quad (A_0 + A_1)B_0 \quad (A_0 + A_1)^2B_0 \quad (A_0 + A_1)^3B_0] = 3 < 4$ and $rank[B_0 \quad A_0B_0 \quad A_0^2B_0 \quad A_0^3B_0] = 3 < 4$, neither (A, B_0) nor $(A + A_1, B_0)$ is controllable. Choosing $\delta = 0.1$ and using LMI toolbox in Matlab, the maximum value of τ_1 is found 0.14. Therefore the system is 0.1-stabilizable with $\tau_1 = 0.14$. The feedback control is $u(t) = Kx(t)$ where $K = [-0.2737 \quad -0.1137 \quad -0.1923 \quad -1.8416]$.

Remark 2: In case $B_0 = B_1 = \cdots = B_m = 0$, a δ-stability criterion for system (1.1) is given in the following corollary:

Corollary 2.3. *System* (1.1) *with* $B_0 = B_1 = \cdots = B_m = 0$ *is δ-stable if there exist positive definite symmetric matrices* $X, W, F_{11}, F_{22}, F_{33}$ *and any matrices* , Y_k, $k = 1, 2, ..., m$, F_{12}, F_{13}, F_{23}, *a number* $\alpha > 1$ *such that* (2.3),(2.4) *hold and the following LMI hold*

$$\Xi(0, 0, ..., 0) < 0. \tag{2.21}$$

The following numerical example shows that this δ-stability criterion may be less conservative than those in the literature.

Example 2.4. Consider the linear systems

$$\begin{cases} \dot{x}(t) & = A_0x(t) + A_1x(t-1) + A_2x(t-\sqrt{3}), \quad t \geq 0 \\ x(t) & = \phi(t), \quad t \in [-\sqrt{3}, 0], \end{cases} \tag{2.22}$$

where $A_0 = \begin{pmatrix} -7 & -1 \\ 0.5 & -5.5 \end{pmatrix}$, $A_1 = \begin{pmatrix} 10.1 & 14.2 \\ -6.6 & -10.2 \end{pmatrix}$, $A_2 \begin{pmatrix} -5.2 & -6.2 \\ 3.6 & 5.1 \end{pmatrix}$.
Using LMI toolbox, the maximum of δ is 0.117.

Table 1: *Comparison between results of our result and recent ones*

Method	[10]	[7]	[6]	Corollary 2.3
δ	0.95	0.1	0.111	0.117

Another interesting problem, which has been introduced in [11], is to find sufficient condition for stabilizability of system

$$\begin{cases} \dot{x}(t) &= A_0x(t) + \sum_{k=1}^{m} A_kx(t-\tau_k) + B_0u(t) \\ x(t) &= \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{2.23}$$

with an instantaneous feedback control term $u(t) = K_0x(t) + \sum_{k=1}^{m} K_kx(t-\tau_k)$. By the same way, we also obtain a δ-stabilizability criterion for system (2.23):

Theorem 2.5. *If there exist positive definite symmetric matrices* $X, W,$ F_{11}, F_{22}, F_{33} *and any matrices* Z_0, Z_k, Y_k*,* $k = 1, 2, ..., m$*,* F_{12}, F_{13}, F_{23}*, a number* $\alpha > 1$ *such that* (2.3),(2.4) *hold and the following LMI hold*

$$\Xi(B_0Z_0, B_0Z_1, ..., B_0Z_m) < 0, \tag{2.24}$$

then system (2.2) *is* δ*-stabilizable with the feedback control*

$$u(t) = Z_0X^{-1}x(t) + Z_1X^{-1}x(t-\tau_1) + \cdots Z_mX^{-1}x(t-\tau_m).$$

Example 2.6. *Consider system* (2.23) *with* $m = 1$*,* $\tau_1 = 5$ *and*

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 05 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.3 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The following table will show that the delayed feedback controller in Theorem 2.5 allows larger convergence rate memoryless controllers. Table 2:

Method	Kubo et. al [4]	Theorem 2.1	Theorem 2.5
δ	0.2	0,45	0.5

3 Conclusions

In this paper, the exponential stabilzability with a given convergence rate has been discussed for linear systems with multiple delays on states and controls. Based on the Lyapunov method and a parameterized neutral model transformation, exponential stabilizability criteria with memoryless feedback control or instantaneous feedback control are proposed. Some numerical examples are given to illustrate proposed criteria.

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