Energy and Some Hamiltonian Properties of Graphs

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Abstract

Using the energy of graphs, we present sufficient conditions for some Hamiltonian properties of graphs.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in [2]. For a graph $G = (V,E)$, $n := |V|$, $e := |E|$, and $G^c := (V,E^c)$, where $E^c := \{xy : x \in V, y \in V, x \neq y, xy \notin E\}$. For a bipartite graph $G_{BPT} = (X,Y;E)$, $G_{BPT}^c := (X,Y;E^c)$, where $E^c := \{xy : x \in X, y \in Y, xy \notin E\}$. The degree of vertex $v_i$ is denoted by $d_i$. The concept of closure of a general graph $G$ was introduced by Bondy and Chvátal [1]. The $k$ - closure of a graph $G$, denoted $cl_k(G)$, is a graph obtained from $G$ by recursively joining two nonadjacent vertices such that their degree sum is at least $k$. The idea for the closure of a balanced bipartite graph can be found in [1] and [6]. The $k$ - closure of a balanced bipartite graph $G_{BPT} = (X,Y;E)$, where $|X| = |Y|$, denoted $cl_k(G_{BPT})$, is a graph obtained from $G$ by recursively joining two nonadjacent vertices $x \in X$ and $y \in Y$ such that their degree sum is at least $k$. We use $C(n,r)$ to denote the number of $r$ - combinations of a set with $n$ distinct elements.

Let $A(G)$ be the adjacency matrix of a graph $G$ of order $n$ and let $\mu_1(A(G)) \leq \mu_2(A(G)) \leq ... \leq \mu_n(A(G))$ be its eigenvalues. Set $\mu_i(G) := \mu_i(A(G))$, $i = 1, 2, ..., n$. Then $\mu_1(G) \leq \mu_2(G) \leq ... \leq \mu_n(G)$ are called the eigenvalues of the graph $G$. The energy, denoted $E(G)$, of a graph $G$, is defined as
$E(G) := \sum_{i=1}^{n} |\mu_i(G)|.$

Fiedler and Nikiforov in [5] recently proved the following theorem.

**Theorem 1** ([5]) Let $G$ be a graph of order $n$.

[1] If $\mu_n(G^c) \leq \sqrt{n - 1}$, then $G$ contains a Hamiltonian path unless $G = K_{n-1} + v$, where $K_{n-1} + v$ is defined as a graph that consists of a complete graph of order $n - 1$ together with an isolated vertex $v$.

[2] If $\mu_n(G^c) \leq \sqrt{n - 2}$, then $G$ contains a Hamiltonian cycle unless $G = K_{n-1} + e$, where $K_{n-1} + e$ is defined as a graph that consists of a complete graph of order $n - 1$ together with a pendent edge $e$.

In this note, we will present theorems involving energy of graphs on some Hamiltonian properties of graphs. Some ideas and techniques developed by Fiedler and Nikiforov in [5] will be used in our proofs.

**Theorem 2** Let $G$ be a graph of order $n \geq 3$. Then $G$ contains a Hamiltonian cycle if

$$\sqrt{(n - 1)e(G^c)}(\sqrt{n + 1} + 1) + 2e(G^c) - E(G^c) < 2n - 4.$$  

**Theorem 3** Let $G$ be a graph of order $n \geq 4$. Then $G$ contains a Hamiltonian path if

$$\sqrt{e(G^c)(\sqrt{n - 1} + 1) + 2e(G^c) - E(G^c)} < 2n - 2.$$  

**Theorem 4** Let $G_{BPT} = (X, Y; E)$, where $|X| = |Y| = r \geq 2$, be a balanced bipartite graph of order $n = 2r \geq 4$. Then $G_{BPT}$ contains a Hamiltonian cycle if

$$\sqrt{e(G_{BPT}^c)(\sqrt{n - 2} + \sqrt{2}) + 2e(G_{BPT}^c) - E(G_{BPT}^c)} < 2r - 2.$$  

2. **Lemmas**

We need the following results as lemmas to prove our theorems.

**Lemma 1** ([1]) A graph $G$ of order $n$ has a Hamiltonian cycle if and only if $cl_n(G)$ has one.

**Lemma 2** ([1]) A graph $G$ of order $n$ has a Hamiltonian path if and only if $cl_{n-1}(G)$ has one.
Lemma 3 ([3]) Let $e$ be any edge in a graph $G$. then $E(G) - 2 \leq E(G - \{ e \}) \leq E(G) + 2$.

Lemma 4 ([6]) A balanced bipartite graph $G_{BPT} = (X, Y; E)$, where $|X| = |Y| = r \geq 2$, has a Hamiltonian cycle if and only if $cl_{r+1}(G_{BPT})$ has one.

3. Proofs

Proof of Theorem 2. Let $G$ be a graph satisfying the conditions in Theorem 2 and $G$ does not have a Hamiltonian cycle. Then $H := cl_n(G)$ does not have a Hamiltonian cycle and therefore $H$ is not $K_n$. Thus there exist two vertices $x$ and $y$ in $V(H)$ such that $xy \not\in E(H)$ and for any pair of nonadjacent vertices $u$ and $v$ in $V(H)$ we have $d_H(u) + d_H(v) \leq n - 1$. Hence for any pair of adjacent vertices $u$ and $v$ in $V(H^c)$ we have that $d_{H^c}(u) + d_{H^c}(v) = n - 1 - d_H(u) + n - 1 - d_H(v) \geq n - 1$. So

$$\sum_{uv \in E(H^c)} d_{H^c}(u) + d_{H^c}(v) \geq (n - 1)e(H^c).$$

Moreover,

$$\sum_{v \in V(H^c)} d_H^2(v) = \sum_{uv \in E(H^c)} d_{H^c}(u) + d_{H^c}(v) \geq (n - 1)e(H^c).$$

From the inequality of Hofmeister [4], we have that $n\mu_n^2(H^c) \geq (n - 1)e(H^c)$, i.e.,

$$\mu_n(H^c) \geq \sqrt{\frac{(n - 1)e(H^c)}{n}}.$$

From the definition of $E(H^c)$ and Cauchy - Schwartz inequality, we have that

$$E(H^c) = \sum_{i=1}^n |\mu_i(H^c)| \leq \mu_n(H^c) + \sqrt{(n - 1)\sum_{i=1}^{n-1} \mu_i^2(H^c)}$$

$$= \mu_n(H^c) + \sqrt{(n - 1)(\sum_{i=1}^n \mu_i^2(H^c) - \mu_n^2(H^c))}$$

$$= \mu_n(H^c) + \sqrt{(n - 1)(2e(H^c) - \mu_n^2(H^c))}.$$

Now consider the function $f(x) = x + \sqrt{(n - 1)(2e(H^c) - x^2)}$. It can be easily verified that $f(x)$ is monotonously decreasing when $\sqrt{\frac{2e(H^c)}{n}} \leq x \leq \mu_n(H^c)$. 


\[ \sqrt{2e(H^c)}. \]

Notice that
\[ \frac{\sqrt{2e(H^c)}}{n} \leq \sqrt{\frac{(n-1)e(H^c)}{n}} \leq \mu_n(H^c) \leq \sqrt{2e(H^c)}. \]

Hence
\[ E(H^c) \leq f(\mu_n(H^c)) \leq f\left(\sqrt{\frac{(n-1)e(H^c)}{n}}\right) = \sqrt{\frac{(n-1)e(H^c)}{n}}(\sqrt{n+1} + 1). \]

Set \( r := e(H) - e(G) \). Then \( e(G^c) - e(H^c) = (C(n, 2) - e(G)) - (C(n, 2) - e(H)) = r \). Since \( H \) has two nonadjacent vertices \( x \) and \( y \) such that \( d_H(x) + d_H(y) \leq n - 1 \), then \( e(H) \leq (n-1) + C(n-2, 2) = (n^2 - 3n + 4)/2 \). Hence \( r \leq (n^2 - 3n + 4)/2 - e(G) = (n^2 - 3n + 4)/2 - (C(n, 2) - e(G^c)) = e(G^c) - n + 2. \)

From Lemma 3, we have that \( E(H^c) \geq E(G^c) - 2r \). Thus \( E(H^c) \geq E(G^c) - 2e(G^c) + 2n - 4 \). Since \( e(H^c) \leq e(G^c) \), we have that
\[ E(G^c) - 2e(G^c) + 2n - 4 \leq \sqrt{\frac{(n-1)e(G^c)}{n}}(\sqrt{n+1} + 1), \]
a contradiction.

**Proof of Theorem 3.** Let \( G \) be a graph satisfying the conditions in Theorem 2 and \( G \) does not have a Hamiltonian path. Then \( H := c_{l_{n-1}}(G) \) does not have a Hamiltonian path and therefore \( H \) is not \( K_n \). Thus there exist two vertices \( u \) and \( v \) in \( V(H) \) such that \( u \neq \emptyset \) \( E(H) \) and for any pair of nonadjacent vertices \( u \) and \( v \) in \( V(H) \) we have \( d_H(u) + d_H(v) \leq n - 2 \). Hence for any pair of adjacent vertices \( u \) and \( v \) in \( V(H^c) \) we have that \( d_{H^c}(u) + d_{H^c}(v) = n - 1 - d_H(u) + n - 1 - d_H(v) \geq n \). Using similar arguments as in Proof of Theorem 2, we can show that
\[ E(H^c) \leq \sqrt{e(G^c)}(\sqrt{n+1} + 1). \]
\[ E(H^c) \geq E(G^c) - 2e(G^c) + 2n - 2. \]
Therefore, we have that
\[ E(G^c) - 2e(G^c) + 2n - 2 \leq \sqrt{e(G^c)}(\sqrt{n+1} + 1), \]
a contradiction.

**Proof of Theorem 4.** Let \( G_{BPT} = (X, Y; E) \), where \( |X| = |Y| = r \geq 2 \), be a balanced bipartite graph of order \( n = 2r \geq 4 \) satisfying the conditions in Theorem 4 and \( G \) does not have a Hamiltonian cycle. Then \( H_{BPT} := c_{l_{r+1}}(G_{BPT}) \) does not have a Hamiltonian cycle and therefore \( H_{BPT} \) is not \( K_{r,r} \).
Thus there exist a vertex $x \in X$ and a vertex $y \in Y$ such that $xy \notin E(H_{BPT}^c)$ and for any pair of nonadjacent vertices $u \in X$ and $v \in Y$ we have that $d_{H_{BPT}^c}(u) + d_{H_{BPT}^c}(v) \leq r$. Hence in $H_{BPT}^c$ for any pair of adjacent vertices $u \in X$ and $v \in Y$ we have that $d_{H_{BPT}^c}(u) + d_{H_{BPT}^c}(v) = r - d_{H_{BPT}^c}(u) + r - d_{H_{BPT}^c}(v) \geq r$. So

$$\sum_{uv \in E(H_{BPT}^c)} d_{H_{BPT}^c}(u) + d_{H_{BPT}^c}(v) \geq re(H_{BPT}^c).$$

Moreover, we have that

$$\sum_{v \in V(H_{BPT}^c)} d_{H_{BPT}^c}^2(v) = \sum_{uv \in E(H_{BPT}^c)} d_{H_{BPT}^c}(u) + d_{H_{BPT}^c}(v) \geq re(H_{BPT}^c).$$

From the inequality of Hofmeister [4], we have that

$$2r\mu_n^2(H_{BPT}^c) \geq \sum_{v \in V(H_{BPT}^c)} d_{H_{BPT}^c}^2(v) \geq re(H_{BPT}^c).$$

So $\mu_n(H_{BPT}^c) \geq \sqrt{e(H_{BPT}^c)/2}$.

Since $H_{BPT}^c$ is a bipartite graph, $\mu_n(H_{BPT}^c) = -\mu_1(H_{BPT}^c)$. From the definition of $E(H_{BPT}^c)$ and Cauchy - Schwartz inequality, we have that

$$E(H_{BPT}^c) = \sum_{i=1}^{n} |\mu_i(H_{BPT}^c)| \leq 2\mu_n(H_{BPT}^c) + \sqrt{(n - 2) \sum_{i=2}^{n-1} \mu_i^2(H_{BPT}^c)}$$

$$= 2\mu_n(H_{BPT}^c) + \sqrt{(n - 2) (\sum_{i=1}^{n} \mu_i^2(H_{BPT}^c) - 2\mu_n^2(H_{BPT}^c))}$$

$$= 2\mu_n(H_{BPT}^c) + \sqrt{(n - 2) (2e(H_{BPT}^c) - 2\mu_n^2(H_{BPT}^c))}.$$

Now consider the function $f(x) = 2x + \sqrt{(n - 2)(2e(H_{BPT}^c) - 2x^2)}$. It can be easily verified that $f(x)$ is monotonously decreasing when $\sqrt{\frac{2e(H_{BPT}^c)}{n}} \leq x \leq \sqrt{e(H_{BPT}^c)}$. Notice that

$$\sqrt{\frac{2e(H_{BPT}^c)}{n}} \leq \sqrt{\frac{e(H_{BPT}^c)}{2}} \leq \mu_n(H_{BPT}^c) \leq \sqrt{e(H_{BPT}^c)}.$$

Hence

$$E(H_{BPT}^c) \leq f(\mu_n(H_{BPT}^c)) \leq f\left(\sqrt{\frac{e(H_{BPT}^c)}{2}}\right) = \sqrt{e(H_{BPT}^c)}(\sqrt{n - 2} + \sqrt{2}).$$
Set \( s := e(H_{BPT}) - e(G_{BPT}) \). Then \( e(G^c_{BPT}) - e(H^c_{BPT}) = (r^2 - e(G_{BPT})) - (r^2 - e(H_{BPT})) = s \). Since \( H_{BPT} \) has two nonadjacent vertices \( x \in X \) and \( y \in Y \) such that \( d_{H_{BPT}}(x) + d_{H_{BPT}}(y) \leq r \), then \( e(H_{BPT}) \leq (r - 1)^2 + r = r^2 - r + 1 \). Hence \( s \leq r^2 - r + 1 - e(G_{BPT}) = r^2 - r + 1 - (r^2 - e(G^c_{BPT})) = e(G^c_{BPT}) - r + 1 \).

From Lemma 3, we have that \( E(H^c_{BPT}) \geq E(G^c_{BPT}) - 2s \). Thus

\[
E(H^c_{BPT}) \geq E(G^c_{BPT}) - 2e(G^c_{BPT}) + 2r - 2.
\]

Since \( e(H^c_{BPT}) \leq e(G^c_{BPT}) \), we have that

\[
E(G^c_{BPT}) - 2e(G^c_{BPT}) + 2r - 2 \leq \sqrt{e(G^c_{BPT})(\sqrt{n} - 2 + \sqrt{2})},
\]

a contradiction. \( \diamond \)

References


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