

# Energy and Some Hamiltonian Properties of Graphs

Rao Li

Dept. of Mathematical Sciences  
University of South Carolina Aiken  
Aiken, SC 29801, USA  
raol@usca.edu

## Abstract

Using the energy of graphs, we present sufficient conditions for some Hamiltonian properties of graphs.

**Keywords:** Energy of graphs, Hamiltonian property of graphs

## 1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in [2]. For a graph  $G = (V, E)$ ,  $n := |V|$ ,  $e := |E|$ , and  $G^c := (V, E^c)$ , where  $E^c := \{xy : x \in V, y \in V, x \neq y, xy \notin E\}$ . For a bipartite graph  $G_{BPT} = (X, Y; E)$ ,  $G_{BPT}^c := (X, Y; E^c)$ , where  $E^c := \{xy : x \in X, y \in Y, xy \notin E\}$ . The degree of vertex  $v_i$  is denoted by  $d_i$ . The concept of closure of a general graph  $G$  was introduced by Bondy and Chvátal [1]. The  $k$ -closure of a graph  $G$ , denoted  $cl_k(G)$ , is a graph obtained from  $G$  by recursively joining two nonadjacent vertices such that their degree sum is at least  $k$ . The idea for the closure of a balanced bipartite graph can be found in [1] and [6]. The  $k$ -closure of a balanced bipartite graph  $G_{BPT} = (X, Y; E)$ , where  $|X| = |Y|$ , denoted  $cl_k(G_{BPT})$ , is a graph obtained from  $G$  by recursively joining two nonadjacent vertices  $x \in X$  and  $y \in Y$  such that their degree sum is at least  $k$ . We use  $C(n, r)$  to denote the number of  $r$ -combinations of a set with  $n$  distinct elements.

Let  $A(G)$  be the adjacency matrix of a graph  $G$  of order  $n$  and let  $\mu_1(A(G)) \leq \mu_2(A(G)) \leq \dots \leq \mu_n(A(G))$  be its eigenvalues. Set  $\mu_i(G) := \mu_i(A(G))$ ,  $i = 1, 2, \dots, n$ . Then  $\mu_1(G) \leq \mu_2(G) \leq \dots \leq \mu_n(G)$  are called the eigenvalues of the graph  $G$ . The energy, denoted  $\mathbf{E}(G)$ , of a graph  $G$ , is defined as

$$\mathbf{E}(G) := \sum_{i=1}^n |\mu_i(G)|.$$

Fiedler and Nikiforov in [5] recently proved the following theorem.

**Theorem 1** ([5]) Let  $G$  be a graph of order  $n$ .

[1] If  $\mu_n(G^c) \leq \sqrt{n-1}$ , then  $G$  contains a Hamiltonian path unless  $G = K_{n-1} + v$ , where  $K_{n-1} + v$  is defined as a graph that consists of a complete graph of order  $n-1$  together with an isolated vertex  $v$ .

[2] If  $\mu_n(G^c) \leq \sqrt{n-2}$ , then  $G$  contains a Hamiltonian cycle unless  $G = K_{n-1} + e$ , where  $K_{n-1} + e$  is defined as a graph that consists of a complete graph of order  $n-1$  together with a pendent edge  $e$ .

In this note, we will present theorems involving energy of graphs on some Hamiltonian properties of graphs. Some ideas and techniques developed by Fiedler and Nikiforov in [5] will be used in our proofs.

**Theorem 2** Let  $G$  be a graph of order  $n \geq 3$ . Then  $G$  contains a Hamiltonian cycle if

$$\sqrt{\frac{(n-1)e(G^c)}{n}}(\sqrt{n+1}+1) + 2e(G^c) - \mathbf{E}(G^c) < 2n-4.$$

**Theorem 3** Let  $G$  be a graph of order  $n \geq 4$ . Then  $G$  contains a Hamiltonian path if

$$\sqrt{e(G^c)}(\sqrt{n-1}+1) + 2e(G^c) - \mathbf{E}(G^c) < 2n-2.$$

**Theorem 4** Let  $G_{BPT} = (X, Y; E)$ , where  $|X| = |Y| = r \geq 2$ , be a balanced bipartite graph of order  $n = 2r \geq 4$ . Then  $G_{BPT}$  contains a Hamiltonian cycle if

$$\sqrt{e(G_{BPT}^c)}(\sqrt{n-2} + \sqrt{2}) + 2e(G_{BPT}^c) - \mathbf{E}(G_{BPT}^c) < 2r-2.$$

## 2. Lemmas

We need the following results as lemmas to prove our theorems.

**Lemma 1** ([1]) A graph  $G$  of order  $n$  has a Hamiltonian cycle if and only if  $cl_n(G)$  has one.

**Lemma 2** ([1]) A graph  $G$  of order  $n$  has a Hamiltonian path if and only if  $cl_{n-1}(G)$  has one.

**Lemma 3** ([3]) Let  $e$  be any edge in a graph  $G$ . then  $\mathbf{E}(G) - 2 \leq \mathbf{E}(G - \{e\}) \leq \mathbf{E}(G) + 2$ .

**Lemma 4** ([6]) A balanced bipartite graph  $G_{BPT} = (X, Y; E)$ , where  $|X| = |Y| = r \geq 2$ , has a Hamiltonian cycle if and only if  $cl_{r+1}(G_{BPT})$  has one.

### 3. Proofs

**Proof of Theorem 2.** Let  $G$  be a graph satisfying the conditions in Theorem 2 and  $G$  does not have a Hamiltonian cycle. Then  $H := cl_n(G)$  does not have a Hamiltonian cycle and therefore  $H$  is not  $K_n$ . Thus there exist two vertices  $x$  and  $y$  in  $V(H)$  such that  $xy \notin E(H)$  and for any pair of nonadjacent vertices  $u$  and  $v$  in  $V(H)$  we have  $d_H(u) + d_H(v) \leq n - 1$ . Hence for any pair of adjacent vertices  $u$  and  $v$  in  $V(H^c)$  we have that  $d_{H^c}(u) + d_{H^c}(v) = n - 1 - d_H(u) + n - 1 - d_H(v) \geq n - 1$ . So

$$\sum_{uv \in E(H^c)} d_{H^c}(u) + d_{H^c}(v) \geq (n - 1)e(H^c).$$

Moreover,

$$\sum_{v \in V(H^c)} d_{H^c}^2(v) = \sum_{uv \in E(H^c)} d_{H^c}(u) + d_{H^c}(v) \geq (n - 1)e(H^c).$$

From the inequality of Hofmeister [4], we have that  $n\mu_n^2(H^c) \geq (n - 1)e(H^c)$ , i.e.,

$$\mu_n(H^c) \geq \sqrt{\frac{(n - 1)e(H^c)}{n}}.$$

From the definition of  $\mathbf{E}(H^c)$  and Cauchy - Schwartz inequality, we have that

$$\begin{aligned} \mathbf{E}(H^c) &= \sum_{i=1}^n |\mu_i(H^c)| \leq \mu_n(H^c) + \sqrt{(n - 1) \sum_{i=1}^{n-1} \mu_i^2(H^c)} \\ &= \mu_n(H^c) + \sqrt{(n - 1) (\sum_{i=1}^n \mu_i^2(H^c) - \mu_n^2(H^c))} \\ &= \mu_n(H^c) + \sqrt{(n - 1)(2e(H^c) - \mu_n^2(H^c))}. \end{aligned}$$

Now consider the function  $f(x) = x + \sqrt{(n - 1)(2e(H^c) - x^2)}$ . It can be easily verified that  $f(x)$  is monotonously decreasing when  $\sqrt{\frac{2e(H^c)}{n}} \leq x \leq$

$\sqrt{2e(H^c)}$ . Notice that

$$\sqrt{\frac{2e(H^c)}{n}} \leq \sqrt{\frac{(n-1)e(H^c)}{n}} \leq \mu_n(H^c) \leq \sqrt{2e(H^c)}.$$

Hence

$$\mathbf{E}(H^c) \leq f(\mu_n(H^c)) \leq f\left(\sqrt{\frac{(n-1)e(H^c)}{n}}\right) = \sqrt{\frac{(n-1)e(H^c)}{n}}(\sqrt{n+1} + 1).$$

Set  $r := e(H) - e(G)$ . Then  $e(G^c) - e(H^c) = (C(n, 2) - e(G)) - (C(n, 2) - e(H)) = r$ . Since  $H$  has two nonadjacent vertices  $x$  and  $y$  such that  $d_H(x) + d_H(y) \leq n - 1$ , then  $e(H) \leq (n - 1) + C(n - 2, 2) = (n^2 - 3n + 4)/2$ . Hence  $r \leq (n^2 - 3n + 4)/2 - e(G) = (n^2 - 3n + 4)/2 - (C(n, 2) - e(G^c)) = e(G^c) - n + 2$ .

From Lemma 3, we have that  $\mathbf{E}(H^c) \geq \mathbf{E}(G^c) - 2r$ . Thus  $\mathbf{E}(H^c) \geq \mathbf{E}(G^c) - 2e(G^c) + 2n - 4$ . Since  $e(H^c) \leq e(G^c)$ , we have that

$$\mathbf{E}(G^c) - 2e(G^c) + 2n - 4 \leq \sqrt{\frac{(n-1)e(G^c)}{n}}(\sqrt{n+1} + 1),$$

a contradiction. ◇

**Proof of Theorem 3.** Let  $G$  be a graph satisfying the conditions in Theorem 2 and  $G$  does not have a Hamiltonian path. Then  $H := cl_{n-1}(G)$  does not have a Hamiltonian path and therefore  $H$  is not  $K_n$ . Thus there exist two vertices  $x$  and  $y$  in  $V(H)$  such that  $xy \notin E(H)$  and for any pair of nonadjacent vertices  $u$  and  $v$  in  $V(H)$  we have  $d_H(u) + d_H(v) \leq n - 2$ . Hence for any pair of adjacent vertices  $u$  and  $v$  in  $V(H^c)$  we have that  $d_{H^c}(u) + d_{H^c}(v) = n - 1 - d_H(u) + n - 1 - d_H(v) \geq n$ . Using similar arguments as in Proof of Theorem 2, we can show that

$$\mathbf{E}(H^c) \leq \sqrt{e(G^c)}(\sqrt{n-1} + 1).$$

$$\mathbf{E}(H^c) \geq \mathbf{E}(G^c) - 2e(G^c) + 2n - 2.$$

Therefore, we have that

$$\mathbf{E}(G^c) - 2e(G^c) + 2n - 2 \leq \sqrt{e(G^c)}(\sqrt{n-1} + 1),$$

a contradiction. ◇

**Proof of Theorem 4.** Let  $G_{BPT} = (X, Y; E)$ , where  $|X| = |Y| = r \geq 2$ , be a balanced bipartite graph of order  $n = 2r \geq 4$  satisfying the conditions in Theorem 4 and  $G$  does not have a Hamiltonian cycle. Then  $H_{BPT} := cl_{r+1}(G_{BPT})$  does not have a Hamiltonian cycle and therefore  $H_{BPT}$  is not  $K_{r,r}$ .

Thus there exist a vertex  $x \in X$  and a vertex  $y \in Y$  such that  $xy \notin E(H_{BPT})$  and for any pair of nonadjacent vertices  $u \in X$  and  $v \in Y$  we have  $d_{H_{BPT}}(u) + d_{H_{BPT}}(v) \leq r$ . Hence in  $H_{BPT}^c$  for any pair of adjacent vertices  $u \in X$  and  $v \in Y$  we have that  $d_{H_{BPT}^c}(u) + d_{H_{BPT}^c}(v) = r - d_{H_{BPT}}(u) + r - d_{H_{BPT}}(v) \geq r$ . So

$$\sum_{uv \in E(H_{BPT}^c)} d_{H_{BPT}^c}(u) + d_{H_{BPT}^c}(v) \geq re(H_{BPT}^c).$$

Moreover, we have that

$$\sum_{v \in V(H_{BPT}^c)} d_{H_{BPT}^c}^2(v) = \sum_{uv \in E(H_{BPT}^c)} d_{H_{BPT}^c}(u) + d_{H_{BPT}^c}(v) \geq re(H_{BPT}^c).$$

From the inequality of Hofmeister [4], we have that

$$2r\mu_n^2(H_{BPT}^c) \geq \sum_{v \in V(H_{BPT}^c)} d_{H_{BPT}^c}^2(v) \geq re(H_{BPT}^c).$$

So  $\mu_n(H_{BPT}^c) \geq \sqrt{e(H_{BPT}^c)/2}$ .

Since  $H_{BPT}^c$  is a bipartite graph,  $\mu_n(H_{BPT}^c) = -\mu_1(H_{BPT}^c)$ . From the definition of  $\mathbf{E}(H_{BPT}^c)$  and Cauchy - Schwartz inequality, we have that

$$\begin{aligned} \mathbf{E}(H_{BPT}^c) &= \sum_{i=1}^n |\mu_i(H_{BPT}^c)| \leq 2\mu_n(H_{BPT}^c) + \sqrt{(n-2) \sum_{i=2}^{n-1} \mu_i^2(H_{BPT}^c)} \\ &= 2\mu_n(H_{BPT}^c) + \sqrt{(n-2) \left( \sum_{i=1}^n \mu_i^2(H_{BPT}^c) - 2\mu_n^2(H_{BPT}^c) \right)} \\ &= 2\mu_n(H_{BPT}^c) + \sqrt{(n-2)(2e(H_{BPT}^c) - 2\mu_n^2(H_{BPT}^c))}. \end{aligned}$$

Now consider the function  $f(x) = 2x + \sqrt{(n-2)(2e(H_{BPT}^c) - 2x^2)}$ . It can be easily verified that  $f(x)$  is monotonously decreasing when  $\sqrt{\frac{2e(H_{BPT}^c)}{n}} \leq x \leq \sqrt{e(H_{BPT}^c)}$ . Notice that

$$\sqrt{\frac{2e(H_{BPT}^c)}{n}} \leq \sqrt{\frac{e(H_{BPT}^c)}{2}} \leq \mu_n(H_{BPT}^c) \leq \sqrt{e(H_{BPT}^c)}.$$

Hence

$$\mathbf{E}(H_{BPT}^c) \leq f(\mu_n(H_{BPT}^c)) \leq f\left(\sqrt{\frac{e(H_{BPT}^c)}{2}}\right) = \sqrt{e(H_{BPT}^c)}(\sqrt{n-2} + \sqrt{2}).$$

Set  $s := e(H_{BPT}) - e(G_{BPT})$ . Then  $e(G_{BPT}^c) - e(H_{BPT}^c) = (r^2 - e(G_{BPT})) - (r^2 - e(H_{BPT})) = s$ . Since  $H_{BPT}$  has two nonadjacent vertices  $x \in X$  and  $y \in Y$  such that  $d_{H_{BPT}}(x) + d_{H_{BPT}}(y) \leq r$ , then  $e(H_{BPT}) \leq (r-1)^2 + r = r^2 - r + 1$ . Hence  $s \leq r^2 - r + 1 - e(G_{BPT}) = r^2 - r + 1 - (r^2 - e(G_{BPT}^c)) = e(G_{BPT}^c) - r + 1$ .

From Lemma 3, we have that  $\mathbf{E}(H_{BPT}^c) \geq \mathbf{E}(G_{BPT}^c) - 2s$ . Thus

$$\mathbf{E}(H_{BPT}^c) \geq \mathbf{E}(G_{BPT}^c) - 2e(G_{BPT}^c) + 2r - 2.$$

Since  $e(H_{BPT}^c) \leq e(G_{BPT}^c)$ , we have that

$$\mathbf{E}(G_{BPT}^c) - 2e(G_{BPT}^c) + 2r - 2 \leq \sqrt{e(G_{BPT}^c)}(\sqrt{n-2} + \sqrt{2}),$$

a contradiction.  $\diamond$

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