The Spectral Moments and Energy of Graphs

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Abstract
The energy of a graph is defined as the sum of the absolute values of its eigenvalues. In this paper, we obtain an upper bound for the energy of a graph that involves its moments.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in [1]. Throughout this paper, $G$ will be always a graph of order $n$ and size $m$. We use $V(G) := \{v_1, v_2, ... v_n\}$ to denote the vertex set of $G$ and $d(v_i)$ or $d_i$, where $1 \leq i \leq n$, to denote the degree of vertex $v_i$. For each $1 \leq i \leq n$, the 2-degree of $v_i$, denoted $t(v_i)$ or $t_i$, is defined as the sum of degrees of the vertices adjacent to $v_i$, the average degree of $v_i$ is defined as $t_i/d_i$, and $\sigma(v_i)$ or $\sigma_i$ is defined as the sum of the 2-degrees of vertices adjacent to $v_i$. We define $\Sigma_k(G)$ as $\Sigma_{i=1}^n d_i^k$. A bipartite graph $G = (X, Y; E)$ is $(a, b)$-semiregular if there exist two constants $a$ and $b$ such that each vertex in $X$ has degree $a$ and each vertex in $Y$ has degree $b$. A bipartite graph $G = (X, Y; E)$ is $(p_x, p_y)$-pseudo-semiregular if there exist two constants $p_x$ and $p_y$ such that each vertex in $X$ has average degree $p_x$ and each vertex in $Y$ has average degree $p_y$. The eigenvalues $\mu_1(G) \geq \mu_2(G) \geq ... \geq \mu_n(G)$ of the adjacency matrix $A(G)$ of $G$ are called the eigenvalues of the graph $G$. The $k$th-spectral moments, denoted $M_k(G)$ or $M_k$, of $G$ is defined as $\Sigma_{i=1}^n \mu_i^k$ (see [6]). The energy of a graph $G$, denoted $E(G)$, is defined as $E(G) = \sum_{i=1}^n |\mu_i|$. This concept was introduced by Gutman in [4] and more information and background on the energy of graphs can be found in [5]. In 1971, McClelland [11] proved that $E(G) \leq \sqrt{2mn}$, the first upper bound for $E(G)$. Since then, more upper bounds for $E(G)$ have
been found and some of them can be found in the following theorems.

**Theorem 1** [8]. Let $G$ be a graph on $n$ vertices and $m$ edges. If $2m \geq n$, then the inequality

$$E(G) \leq \frac{2m}{n} + \sqrt{(n-1)[2m - \left(\frac{2m}{n}\right)^2]}$$  \hspace{1cm} (1)

holds. Moreover, equality holds if and only if $G$ is either $\frac{n}{2}K_2$, $K_n$, or a non-complete connected strongly regular graph with two non-trivial eigenvalues with absolute value $\sqrt{(2m - \left(\frac{2m}{n}\right)^2)/(n-1)}$. If $2m \leq n$, then the inequality $E(G) \leq 2m$ holds. Moreover, equality holds if and only if $G$ is disjoint union of edges and isolated vertices.

**Theorem 2** [9]. If $2m \geq n$ and $G$ is a bipartite graph with $n > 2$ vertices and $m$ edges, then the inequality

$$E(G) \leq 2\left(\frac{2m}{n}\right) + \sqrt{(n-2)[2m - 2\left(\frac{2m}{n}\right)^2]}$$  \hspace{1cm} (2)

holds. Moreover, equality holds if and only if at least one of the following statements holds:

1. $n = 2m$ and $G = mK_2$.
2. $n = 2t$, $m = t^2$, and $G = K_{t,t}$.
3. $n = 2\nu$, $2\sqrt{m} < n < 2m$, and $G$ is the incidence graph of a symmetric $2 - (\nu, k, \lambda)$-design with $k = \frac{2m}{n}$ and $\lambda = \frac{k(k-1)}{\nu-1}$.

Theorem 1 and Theorem 2 were generalized by several authors (see [13] [12]) and the latest ones are the following Theorem 3 and Theorem 4 proved by Liu, Lu, and Tian.

**Theorem 3** [10]. Let $G$ be a non-empty graph on $n$ vertices, $m$ edges. Then the inequality

$$E(G) \leq \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} \sigma_i^2}} + \sqrt{(n-1)[2m - \frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} \sigma_i^2}]}$$  \hspace{1cm} (3)

holds. Moreover, equality in (3) holds if and only if $G$ is either $\frac{n}{2}K_2$, $K_n$, or a non-bipartite connected graph satisfying $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n}$ and has three distinct eigenvalues $(p, \sqrt{\frac{2m-p^2}{n-1}}, -\sqrt{\frac{2m-p^2}{n-1}})$, where $p = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n} > \sqrt{\frac{2m}{n}}$. 


Theorem 4 [10]. Let $G = (X, Y)$ be a non-empty bipartite graph with $n > 2$ vertices and $m$ edges. Then the inequality
\[
E(G) \leq 2 \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} + \sqrt{(n - 2)(2m - 2 \sum_{i=1}^{n} \sigma_i^2)}
\]  
(4)
holds. Moreover, equality in (4) holds if and only if $G$ is either $\frac{n}{2}K_2$, $K_{r_1,r_2} \cup (n - r_1 - r_2)K_1$, where $r_1r_2 = m$, or a connected bipartite graph with $V = \{v_1, v_2, ..., v_s\} \cup \{v_{s+1}, v_{s+2}, ..., v_n\}$ such that $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$ and $\frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$, and has four distinct eigenvalues ($\sqrt{pxpy}$, $\sqrt{\frac{2m - 2pxpy}{n - 2}}$, $-\sqrt{\frac{2m - 2pxpy}{n - 2}}$, $-\sqrt{pxpy}$), where $p_x = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$, $p_y = \frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$ and $\sqrt{pxpy} > \frac{2m}{n}$.

Motivated by Theorem 3 and Theorem 4 above, we in this paper prove the following theorems.

Theorem 5. Let $G$ be a non-empty graph on $n$ vertices. If $\sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} \geq (\frac{M_{2k}}{n})^{\frac{1}{2k}}$, where $k$ is a positive integer, then the inequality
\[
E(G) \leq \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} + (n - 1)^{\frac{2k+1}{2k}} (M_{2k} - (\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2})^k)^{\frac{1}{2k}}
\]  
(5)
holds. Moreover, equality in (5) holds if and only if $G$ is either $\frac{n}{2}K_2$, $K_n$, or a non-bipartite connected graph satisfying $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n}$ and has three distinct eigenvalues $(p, (\frac{M_{2k} - p_2k}{n-1})^{\frac{1}{2k}}, (\frac{M_{2k} - p_2k}{n-1})^{\frac{1}{2k}})$, where $p = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n}$.

Theorem 6. Let $G = (X, Y)$ be a non-empty bipartite graph with $n > 2$ vertices and $m$ edges. If $\sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} \geq (\frac{M_{2k}}{n})^{\frac{1}{2k}}$, where $k$ is a positive integer, then the inequality
\[
E(G) \leq 2 \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} + (n - 2)^{\frac{2k-1}{2k}} (M_{2k} - 2(\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2})^k)^{\frac{1}{2k}}
\]  
(6)
holds. Moreover, equality in (6) holds if and only if $G$ is either $\frac{n}{2}K_2$, $K_{r_1,r_2} \cup (n - r_1 - r_2)K_1$, where $r_1r_2 = m$, or a connected bipartite graph with $V = \{v_1, v_2, ..., v_s\} \cup \{v_{s+1}, v_{s+2}, ..., v_n\}$ such that $\frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$ and $\frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$, and has four distinct eigenvalues ($\sqrt{pxpy}$, $(\frac{M_{2k} - 2(\frac{pxpy}{n-2})}{n-2})^{\frac{1}{2k}}$, $-\frac{M_{2k} - 2(\frac{pxpy}{n-2})}{n-2}^{\frac{1}{2k}}$, $-\sqrt{pxpy}$), where $p_x = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_s}{t_s}$, $p_y = \frac{\sigma_{s+1}}{t_{s+1}} = \frac{\sigma_{s+2}}{t_{s+2}} = \cdots = \frac{\sigma_n}{t_n}$.
Lemma 1′. Let \( G \) be a non-empty simple graph of order \( n \). Then

\[
\frac{\sigma_{i+1}}{t_{i+1}} = \cdots = \frac{\sigma_n}{t_n} \quad \text{and} \quad \frac{1}{\sqrt{\mu_1}} = \frac{(M_2/n)^{\frac{1}{n}}}{\mu_1}.
\]

Notice that \( \frac{\sum_{i=1}^{n} \sigma_i}{\sum_{i=1}^{n} t_i} \geq (M_2/n)^{\frac{1}{n}} \) is always true when \( k = 1 \) (see [10]) and \( M_2 = 2m \). Thus if we let \( k = 1 \) in Theorem 5 and Theorem 6, then they become Theorem 3 and Theorem 4 respectively.

Let \( q \) be the number of quadrangles in a graph \( G \). Then \( M_4(G) = 2 \sum_{i=1}^{n} d_i^2 - 2m + 8q \) (see [6]). Thus if we let \( k = 2 \) and replace \( M_4 \) by \( 2 \sum_{i=1}^{n} d_i^2 - 2m + 8q \) in Theorem 5 and Theorem 6 then we can obtain upper bounds for general graphs and bipartite graphs which satisfy respectively the conditions in Theorem 5 and Theorem 6.

2. Lemmas

In order to prove Theorem 5 and Theorem 6, we need the following results as our lemmas. The first one is a theorem proved by Hong and Zhang in [7].

**Lemma 1** [7]. Let \( G \) be a simple connected graph of order \( n \). Then

\[
\mu_1 \geq \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}}
\]

with equality if and only if \( \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n} \) or \( G \) is a bipartite graph with \( V = \{v_1, v_2, \ldots, v_s\} \cup \{v_{s+1}, v_{s+2}, \ldots, v_n\} \) such that \( \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n} \) and \( \frac{\sigma_{i+1}}{t_{i+1}} = \frac{\sigma_{i+2}}{t_{i+2}} = \cdots = \frac{\sigma_n}{t_n} \).

In fact, the above Hong and Zhang’s theorem can be slightly strengthened to the following Lemma 1′. Notice that Lemma 1′ has been used by Liu, Lu, and Tian in [10] to obtain their results.

**Lemma 1′.** Let \( G \) be a non-empty simple graph of order \( n \). Then

\[
\mu_1 \geq \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}}
\]

with equality if and only if \( \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n} \) with \( \mu_1 = \frac{\sigma_1}{t_1} \) or \( G \) is a bipartite graph with \( V = \{v_1, v_2, \ldots, v_s\} \cup \{v_{s+1}, v_{s+2}, \ldots, v_n\} \) such that \( \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n} \) and \( \frac{\sigma_{i+1}}{t_{i+1}} = \frac{\sigma_{i+2}}{t_{i+2}} = \cdots = \frac{\sigma_n}{t_n} \) with \( \mu_1 = \sqrt{p_xp_y} \), where \( p_x = \frac{\sigma_1}{t_1} \) and \( p_y = \frac{\sigma_n}{t_n} \).

The following Lemma 2 and Lemma 3 can be found in [3] and [2] respectively.
Lemma 2 [3]. A graph $G$ has only one distinct eigenvalue if and only if $G$ is an empty graph. A graph $G$ has two distinct eigenvalues $\mu_1 > \mu_2$ with multiplicities $s_1$ and $s_2$ if and only if $G$ is the direct sum of $s_1$ complete graphs of order $\mu_1 + 1$. In this case, $\mu_2 = -1$ and $s_2 = s_1 \mu_1$.

Lemma 3 [2]. Let $G$ be a graph with $m$ edges. Then $E(G) \geq 2\sqrt{m}$ with equality if and only if $G$ is a complete bipartite graph plus arbitrarily many isolated vertices.

3. Proofs

Proof of Theorem 5. Let $G$ be a graph satisfying the conditions in Theorem 9. Set $\alpha := \frac{1}{2k}$ and $\beta := 1 - \alpha$. By the Hölder inequality, we have that

$$\sum_{i=2}^{n} |\mu_i| \leq \left( \sum_{i=2}^{n} \frac{1}{\beta} \right)^{\alpha} \left( \sum_{i=2}^{n} |\mu_i|^\beta \right)^{1-\alpha}.$$ 

Therefore

$$E(G) = \sum_{i=1}^{n} |\mu_i| \leq \mu_1 + (n-1)^\beta (M_{2k} - \mu_{1}^{2k})^{\frac{1}{2k}}.$$ 

Consider the function $f(x) = x + (n-1)^\beta (M_{2k} - x^{2k})^{\frac{1}{2k}}$. It can be verified that $f(x)$ is decreasing when $(\frac{M_{2k}}{n})^{\frac{1}{2k}} \leq x \leq M_{2k}^{\frac{1}{2k}}$.

From Lemma 1' and the assumption that

$$\sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} \geq \left( \frac{M_{2k}}{n} \right)^{\frac{1}{2k}},$$

we have that

$$E(G) \leq f(\mu_1) \leq f\left( \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} \right).$$

Therefore Inequality (5) is proved.

If $G$ is $\frac{n}{2}K_2$, then $\sqrt{\frac{\sum_{i=1}^{n-1} \sigma_i^2}{\sum_{i=1}^{n-1} t_i^2}} = 1 = \left( \frac{M_{2k}}{n} \right)^{\frac{1}{2k}}$ and both sides of Inequality (5) are equal to $n$.

If $G$ is $K_n$, then $\sqrt{\frac{\sum_{i=1}^{n-1} \sigma_i^2}{\sum_{i=1}^{n-1} t_i^2}} = n - 1$. From $(n-1) \geq \left( \frac{(n-1)^{2k}+(n-1)^{1}}{n} \right)^{\frac{1}{2k}}$, we have $\sqrt{\frac{\sum_{i=1}^{n-1} \sigma_i^2}{\sum_{i=1}^{n-1} t_i^2}} = (n-1) \geq \left( \frac{(n-1)^{2k}+(n-1)^{1}}{n} \right)^{\frac{1}{2k}} = \left( \frac{M_{2k}}{n} \right)^{\frac{1}{2k}}$ and both sides of Inequality (5) are equal to $2(n-1)$. 

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Lemma 2 implies that $\mu G_2 \leq s$, the multiplicity we have that $|v_1 \{\mu \text{ and } s, \text{ and has three distinct eigenvalues (} G \text{ cannot be a bipartite graph. From Case 1.} \text{ cases.}} \text{ Since } G \text{ is a non-bipartite connected graph satisfying the conditions in Theorem 10. Then } \mu_1 = -\mu_n. \text{ Hence } G \text{ is } K_n. \text{ Case 2. } G \text{ has three distinct eigenvalues}} \text{ existence of an integer } r \text{ such that } \mu_1 > \mu_2 = \cdots = \mu_r > 0 > \mu_{r+1} = \cdots = \mu_n \text{ and } \mu_2 = -\mu_n. \text{ Hence } G \text{ cannot be a bipartite graph. From } \mu_1 = \sqrt{\frac{\sum_{i=1}^n \sigma_i^2}{\sum_{i=1}^n t_i}} \text{ and } \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_n}{t_n}, \text{ we have that } \mu_1 = \frac{\alpha_1}{t_1} = \frac{\alpha_2}{t_2} = \cdots = \frac{\alpha_n}{t_n}. \text{ Since } \mu_1 > \mu_i, \text{ for each } i \text{ with } 2 \leq i \leq n, \text{ and } G \text{ must be connected. Set } p := \mu_1. \text{ Then } G \text{ has three distinct eigenvalues (} p, (\frac{M_{2k-n-1}}{n-1})^{ \frac{1}{2k}}, - (\frac{M_{2k-n-1}}{n-1})^{ \frac{1}{2k}}), \text{ where } p = \frac{\alpha_1}{t_1} = \frac{\alpha_2}{t_2} = \cdots = \frac{\alpha_n}{t_n} > (\frac{M_{2k}}{n})^{ \frac{1}{2k}}. \text{ Proof of Theorem 6. Let } G \text{ be a graph satisfying the conditions in Theorem 10. Then } \mu_1 = -\mu_n. \text{ Set } \alpha := \frac{1}{2k} \text{ and } \beta := 1 - \alpha. \text{ By the Hölder inequality, we have that}}$ 

$$\sum_{i=2}^{n-1} |\mu_i| \leq \left( \sum_{i=2}^{n-1} |\mu_i|^{\beta} \right)^{\frac{1}{\beta}} \left( \sum_{i=2}^{n-1} |\mu_i|^{\frac{1}{\alpha}} \right)^{\alpha}.$$
Therefore
\[ E(G) = \sum_{i=1}^{n} |\mu_i| \leq 2\mu_1 + (n - 2)^2(M_{2k} - 2\mu_1^{2k})^{\frac{1}{2k}}. \]

Consider the function \( f(x) = 2x + (n - 2)^2(M_{2k} - 2x^{2k})^{\frac{1}{2k}} \). It can be verified that \( f(x) \) is decreasing when \( (\frac{M_{2k}}{n})^{\frac{1}{2k}} \leq x \leq M_{2k}^{\frac{1}{2k}} \).

From Lemma 1' and the assumption that
\[ \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} \geq \left( \frac{M_{2k}}{n} \right)^{\frac{1}{2k}}, \]
we have that
\[ E(G) \leq f(\mu_1) \leq f\left( \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} \right). \]
Therefore Inequality (6) is proved.

If \( G \) is \( \frac{n}{2}K_2 \), then \( \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} = 1 = \left( \frac{M_{2k}}{n} \right)^{\frac{1}{2k}} \) and both sides of Inequality (6) are equal to \( n \).

If \( G \) is \( K_{r_1,r_2} \cup (n - r_1 - r_2)K_1 \), where \( r_1r_2 = m \), then \( \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} = \sqrt{r_1r_2} \geq \left( \frac{2(r_1r_2)^{\frac{k}{2}}}{n} \right)^{\frac{1}{2k}} \) and both sides of Inequality (6) are equal to \( 2\sqrt{r_1r_2} \).

If \( G \) is a connected bipartite graph with \( V = \{v_1, v_2, ..., v_s\} \cup \{ v_{s+1}, v_{s+2}, ..., v_n \} \) such that \( \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_4}{t_4} \) and \( \frac{\sigma_5}{t_5} = \frac{\sigma_6}{t_6} = \cdots = \frac{\sigma_n}{t_n} \), and has four distinct eigenvalues \( (\sqrt{p_xp_y}, (\frac{M_{2k}-2(p_xp_y)}{n-2})^{\frac{1}{2k}}), (\frac{M_{2k}-2(p_xp_y)}{n-2})^{\frac{1}{2k}}, -\sqrt{p_xp_y}) \), where \( p_x = \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_4}{t_4}, p_y = \frac{\sigma_5}{t_5} = \frac{\sigma_6}{t_6} = \cdots = \frac{\sigma_n}{t_n} \) and \( \sqrt{p_xp_y} > (\frac{M_{2k}}{n})^{\frac{1}{2k}} \), then both sides of Inequality (6) are equal to \( 2p + (n - 2)(\frac{M_{2k}-2p^{2k}}{n-2})^{\frac{1}{2k}} \), where \( p = \sqrt{p_xp_y} \).

Now suppose that Inequality (6) becomes an equality. Then we have that \( |\mu_2| = |\mu_3| = \cdots = |\mu_{n-1}| = (\frac{M_{2k}-2\mu_1^{2k}}{n-2})^{\frac{1}{2k}} \) and \( \mu_1 = -\mu_n = \sqrt{\frac{\sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} t_i^2}} \). By Lemma 1', we have that \( \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_4}{t_4} \) or \( G \) is a bipartite graph with \( V = \{v_1, v_2, ..., v_s\} \cup \{ v_{s+1}, v_{s+2}, ..., v_n \} \) such that \( \frac{\sigma_1}{t_1} = \frac{\sigma_2}{t_2} = \cdots = \frac{\sigma_4}{t_4} \) and \( \frac{\sigma_5}{t_5} = \frac{\sigma_6}{t_6} = \cdots = \frac{\sigma_n}{t_n} \) with \( \mu_1 = \sqrt{p_xp_y} \), where \( p_x = \frac{\sigma_1}{t_1} \) and \( p_y = \frac{\sigma_n}{t_n} \). Since \( G \) is a non-empty graph, Lemma 2 implies that \( G \) has at least two distinct eigenvalues. Hence we just have the following possible cases.

**Case 1.** \( G \) has two distinct eigenvalues with the same absolute values
Then Lemma 2 implies that \( \mu_1 = -\mu_n = |\mu_2| = \cdots = |\mu_{n-1}| = 1 \). Since \( \sum_{i=1}^{n} \mu_i = 0 \), the multiplicity \( s_1 \) of \( \mu_1 = 1 \) must be equal to \( \frac{n}{2} \). Hence \( G \) is the direct sum of \( s_1 = \frac{n}{2} \) complete graphs of order \( \mu_1 + 1 = 2 \). Namely, \( G \) is \( \frac{n}{2} K_2 \).

**Case 2.** \( G \) has three distinct eigenvalues

Since \( G \) is a bipartite graph, we must have that \( \mu_1 = -\mu_n \neq 0 \) and \( \mu_2 = \cdots = \mu_{n-1} = 0 \). Thus \( E(G) = 2 \mu_1 \). From Lemma 3, we have that \( 2\mu_1 \geq 2\sqrt{m} \). Thus \( 2\mu_1^2 \geq 2m \). Notice that \( 2m = \sum_{i=1}^{n} \mu_i^2 = 2\mu_1^2 \). Therefore \( \mu_1 = \sqrt{m} \) and \( E(G) = 2\sqrt{m} \). Hence by Lemma 3 \( G \) is a complete bipartite graph plus arbitrarily many isolated vertices. Namely, there exist integers \( r_1 \geq 1 \) and \( r_2 \geq 1 \) such that \( G \) is \( K_{r_1, r_2} \cup (n - r_1 - r_2)K_1 \), where \( r_1r_2 = m \).

**Case 3.** \( G \) has four distinct eigenvalues

Since \( \mu_1 = -\mu_n \), \( |\mu_2| = \cdots = |\mu_{n-1}| \), and \( G \) has four distinct eigenvalues, the multiplicity of \( \mu_1 \) must be one. Hence we have by Lemma 1’ that \( G \) is a connected bipartite graph with \( V = \{ v_1, v_2, \ldots, v_s \} \cup \{ v_{s+1}, v_{s+2}, \ldots, v_n \} \) such that \( \frac{s_1}{t_1} = \frac{s_2}{t_2} = \cdots = \frac{s_e}{t_e} \) and \( \frac{s_{e+1}}{t_{e+1}} = \frac{s_{e+2}}{t_{e+2}} = \cdots = \frac{s_n}{t_n} \), and has four distinct eigenvalues \( \{ \sqrt{p_xp_y}, \frac{(M_{2k-2}(p_xp_y)^k)}{n-2} \} \), \( \{ \sqrt{p_xp_y}, -(M_{2k-2}(p_xp_y)^k) \} \), \( \{ \sqrt{p_xp_y} \} \), and \( \{ \sqrt{p_xp_y} \} \), where \( p_x = \frac{s_1}{t_1} = \frac{s_2}{t_2} = \cdots = \frac{s_e}{t_e} \), \( p_y = \frac{s_{e+1}}{t_{e+1}} = \frac{s_{e+2}}{t_{e+2}} = \cdots = \frac{s_n}{t_n} \) and \( \sqrt{p_xp_y} > \frac{(M_{2k})^{1/k}}{n} \). \( \square \)

**References**


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