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Iterative Entropy Method for Linear Complementarity Problem

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Abstract. Linear complementarity problem can be formulated as a fix point problem.The paper deals with this problem by means of taking the place of maximum entropy funciton.By constructing an interval extension of entropy funciton,a new interval algorithm is presented.The relevant properties are proven.

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1 Introduction

Complementarity theory, which has been studied in the last several decades[1,2,3], is generally considered to be a domain of applied mathematics.The complementarity problem arises in a variety of contexs such as optimization,game theory,economics,classical mechanics,stochastic optimal control,etc.

Let M ba an $n \times n$ matix and $x \in R^n$ .The linear complementarity problem, denoted by LCP(M,q),is to find a vector x such that

$$x \geq 0, (Mx+q) \geq 0, x^T(Mx+q) = 0 . \tag{1.1}$$

The following notations is used throughout our paper.We define the absolution value of a vector x by

$$|x| = (|x_1|, |x_2|, \cdots, |x_n|)^T$$

$X = (X_1, X_2, \cdots, X_n)^T$ is a n-dimensional interval vector,which $X_i = [\underline{x_i}, \overline{x_i}]$ is a closed bounded set.

Let $I(R_+^n)$ be the set of all interval vector in R_+^n.

Let f be a real valued function of n real variables $x_1, x_2, \cdots, x_n$.By an interval extension of f, we mean an interval valued function Fof n interval variables $X_1, X_2, \cdots, X_n$, with the property $F(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n)$ for real variable.The further knowledge about interval mathematics can be found in [4], [5], etc.

2 Maximum Entropy Function for LCP

The problem (1.1) can be formulated as a fix point problem[4].Just as following：

Lemma 2.1[4] x^* is the solution of (1.1) if and only if x^* is a fixed point of the map

$$h(x) = \max\{0, x - (Mx + q)\}. \tag{1.2}$$

Let $N = I - M, r = q$, so (1.2) can be transforted as following

$$g(x) = \max\{0, Nx - r\} \tag{1.3}$$

Let $F(x, p)$ be maximum entropy function [6]of $g(x)$,which

$$F(x, p) = (F_1(x, p), \cdots, F_n(x, p))^T,$$

$$F_i(x, p) = \frac{1}{p}\ln\{1 + \exp(p(\sum_{j=1}^{n} n_{ij}x_j - r_i)))\}, \ (i = 1, 2, \cdots, n)$$

Lemma 2.2 When $p \to +\infty$, $F(x,p)$ is uniform convergence to $g(x)$ in its domain with the following inequality

$$g_i(x) \le F_i(x,p) \le g_i(x) + \frac{1}{p}\ln 2 \qquad (i = 1, 2, \cdots, n)$$

Proof With the expression of $F_i(x,p)$, we get

$$F_i(x,p) - g_i(x) = \frac{1}{p}\ln\{1 + \exp(p(\sum_{j=1}^{n} n_{ij}x_j - r_i))\} - g_i(x)$$

$$= \frac{1}{p}\ln\{\exp(p(0 - g_i(x))) + \exp(p(\sum_{i=1}^{n} n_{ij}x_j - r_i - g_i(x)))\}$$

From $g_i(x) = \max\{0, \sum_{i=1}^{n} n_{ij}x_j - r_i\}$, we have

$-g_i(x) \le 0$ and $\sum_{i=1}^{n} n_{ij}x_j - r_i - g_i(x) \le 0$,also one of two inequality is zero .So we have

$$1 \le \exp(p(-g_i(x))) + \exp(p(\sum_{i=1}^{n} n_{ij}x_j - r_i - g_i(x))) \le 2$$

then , we get

$$g_i(x) \le F_i(x,p) \le g_i(x) + \frac{1}{p}\ln 2 .$$

This completes the proof of the lemma2.2.

Lemma 2.3 $F_i(x,p)$ is monotone decreasing about p , i.e.

$$F_i(x,p_2) > F_i(x,p_1) \quad \text{when } p_1 > p_2 > 1 .(i = 1, 2, \cdots, n)$$

Proof From Jensen inequality

when $p_1 > p_2 > 1$,

$$\{\exp(p_1 \bullet 0) + \exp(p_1(\sum_{j=1}^{n} n_{ij}x_j - r_i))\}^{\frac{1}{p_1}}$$

$$< \{\exp(p_2 \bullet 0) + \exp(p_2(\sum_{j=1}^{n} n_{ij}x_j - r_i))\}^{\frac{1}{p_2}}$$

then ,we have

$$\frac{1}{p_1}\ln\{1 + \exp(p_1(\sum_{j=1}^{n} n_{ij}x_j - r_i))\}$$

$$< \frac{1}{p_2}\ln\{1 + \exp(p_2(\sum_{j=1}^{n} n_{ij}x_j - r_i))\}$$

so $F_i(x, p_2) > F_i(x, p_1)$.

Lemma 2.4 If x_p is a solution of the equation $F(x, p) = x$,then x_p is the approximation solution of （1.2）

Proof Now we consider

$$\begin{aligned} &\left|g(x_p) - x_p\right| \\ &= \left|g(x_p) - F(x_p, p) + F(x_p, p) - x_p\right| \\ &= \left|g(x_p) - F(x_p, p)\right| \end{aligned}$$

from lemma 2.2，about $\forall \varepsilon > 0$ ，$\exists M > 0$ ，satisfies

$$\left|g(x_p) - F(x_p, p)\right| < \varepsilon$$

i.e. $\left|g(x_p) - x_p\right| \le \varepsilon$, then we have for any given $\varepsilon > 0$, x_p is the approximation solution of （1.2）.

From the proceeding result, we know that (1.2) equal the equation

$$F(x, p) = x. \qquad (1.4)$$

3 Interval Extension of Entropy Function

Denote $[\underline{s_i}, \overline{s_i}] = \sum_{j=1}^{n} n_{ij}X_j - r_i \qquad (i = 1, 2, \cdots, n)$

Define interval function

$$T(X,p)=(T_1(X,p),\cdots,T_n(X,p))^T$$

which

$$T_i(X,p)=\frac{1}{p}\ln\{1+\exp(p(\sum_{j=1}^{n}n_{ij}X_j-r_i))\}\quad (i=1,2,\cdots,n)$$

Theorem 3.1 $T(X,p)$ is a interval extension of $F(x,p)$, and exceed width is zero.

Proof For $i=1,2,\cdots,n$,

$$T_i(x,p)=\frac{1}{p}\ln\{1+\exp(p(\sum_{j=1}^{n}n_{ij}x_j-r_i))\}=F_i(x,p)$$

with inclusion monotonic of interval arithmetic, we know $\sum_{j=1}^{n}n_{ij}x_j-r_i\in\sum_{j=1}^{n}n_{ij}X_j-r_i$,

So we get

$$F_i(x,p)=\frac{1}{p}\ln\{1+\exp(p(\sum_{j=1}^{n}n_{ij}x_j-r_i))\}\in\frac{1}{p}\ln\{1+\exp(p(\sum_{j=1}^{n}n_{ij}X_j-r_i))\}$$

$$=T_i(X,p)\ ,$$

then $F(x,p)\in T(X,p)$,so $T(X,p)$ is an interval extension of $F(x,p)$.

Also $\sum_{j=1}^{n}n_{ij}X_j-r_i$ is a natural interval extension of $\sum_{j=1}^{n}n_{ij}x_j-r_i$,and each of the variable occurs only once in $\sum_{j=1}^{n}n_{ij}x_j-r_i$,from lemma1.1 , the exceed width of $\sum_{j=1}^{n}n_{ij}X_j-r_i$ is zero , i.e. $\underline{s_i}=\min_{x\in X}(\sum_{j=1}^{n}n_{ij}x_j-r_i)$, $\overline{s_i}=\max_{x\in X}(\sum_{j=1}^{n}n_{ij}x_j-r_i)$.

Further, $\ln x$ *and* $\exp x$ are the monotone function , we know

$$\overline{F_i}(X,p)=\frac{1}{p}\ln(1+\exp(p\overline{s_i}))=\overline{T_i}(X,p)$$

$$\underline{F_i}(X,p)=\frac{1}{p}\ln(1+\exp(p\underline{s_i}))=\underline{T_i}(X,p)$$

so we get the exceed width of $T(X,p)$ is zero.

Theorem 3.2 For $\forall\ X\in I(R_+^n)$

(1) If $x^*\in X$ is a solution of $F(x,p)=x$,then $x^*\in T(X,p)$.

(2) If $T(X,p)\bigcap X=\varnothing$, then the solution of (1.4) is not exit in X.

(3) If $T(X,p)\subseteq X$, then there is a solution of (1.4) in X.

Proof (1) Let x^* be a solution of (1.4), i.e. $x^*=F(x^*,p)$, from $T(X,p)$ is a interval extension of $F(x,p)$ in X , $x^*\in X$, so $F(x^*,p)\in T(X,p)$, i.e. $x^*\in T(X,p)$。

(2) If there is not a solution of(1.4) in X , from the statement of (1),we get $x^*\in T(X,p)$, this contradict with $T(X,p)\bigcap X=\varnothing$.

(3) For $\forall x\in X$, $F(x,p)\in X$, and X is a bounded convex set, from Brouwer
theorem ,we know there is a fixed point $x^*\in X$ satisfied $x^*=F(x^*,p)$.

4 Algorithm and Convergence

With the above ideas, we suggest the following algorithm for solving problem (1.1)

$$\begin{cases} T_i(X^{(k)}, p) = \dfrac{1}{p}\ln\{1+\exp(p(\sum_{j=1}^{n} n_{ij} X_j^{(k)} - r_i))\} \quad (i = 1, 2, \cdots, n) \\ X^{(k+1)} = T\left(X^{(k)}, p\right) \cap X^{(k)} \quad (k = 0, 1, 2, \cdots) \end{cases}$$

(1.5))

Theorem 4.1 $W(T(X,p)) \le W(NX - r)$.

Proof From $\ln x$ *and* $\exp x$ are the monotone function, we get

$$\overline{T_i}(X,p) - \underline{T_i}(X,p)$$

$$= \frac{1}{p}\ln\{1+\exp(p\overline{s_i})\} - \frac{1}{p}\ln\{1+\exp(p\underline{s_i})\}$$

$$= \frac{1}{p}\frac{\exp(p\xi)}{1+\exp(p\xi)}\, p(\overline{s_i} - \underline{s_i})$$

$$\le \overline{s_i} - \underline{s_i}$$

which $\xi \in [\overline{s_i} - \underline{s_i}]$, so $W(T_i(X,p)) \le W((\sum_{j=1}^{n} n_{ij} X_j - r_i))$, i.e.

$$W(T(X,p)) \le W(NX - r).$$

Theorem 4.2 Let M be a H-matrix, where $m_{ii} > 0, i = 1, 2, \cdots, n$, if there is a solution x_p of (1.4) in $X^{(0)} \in I(R_+^n)$, so (3.7) can get a serial $\{X^k\}$,satisfy $\lim_{k\to\infty} X^{(k)} = x_p$.

Proof From (3.7),

$$X^{(k+1)} \subseteq \frac{1}{p}\ln\{1+\exp(p(NX^{(k)} - r))\}$$

so
$$r(X^{(k+1)}) \le r(\frac{1}{p}\ln\{1+\exp(p(NX^{(k)} - r))\})$$

from theorem 4.1
$$r(X^{(k+1)}) \le r(NX^{(k)} - r)$$

$$\leq r(NX^{(k)}) = r((I - D\overline{M})X^{(k)})$$

$$= (I - D\overline{M})r(X^{(k)})$$

where M is H-matrix , so $\rho(I - D\overline{M}) < 1$, from theorem2.2[61],we get $r(X^{(k)}) \to 0$, then $x_p \in X^{(0)}$, so $x_p \in X^{(k)}$.

The detailed steps of the algorithm are the following.

Step 1 Set $X = X^{(0)}$,let p be sufficiently large.

Step 2 Compute $T_i(X, p) = \frac{1}{p}\ln\{1 + \exp(p(\sum_{j=1}^{n} n_{ij}X_j - r_i))\} \quad (i = 1, 2, \cdots, n)$

Step 3 Let $X^{(k+1)} = T\left(X^{(k)}, p\right) \cap X^{(K)}$.

Step 4 If $X^{(k+1)} = \varnothing$, then we terminate the algorithm,;Otherwise proceed with step 2.

Step 5 If $\left\|W(X^{(k+1)})\right\|_\infty < \varepsilon$, then we obtain the solution $X^{(k+1)}$, and terminate the algorithm.Otherwise proceed with step 2.

5 Numerical Results

The algorithm has been implemented using Turbo C2.0 on computer.

$X^{(0)}$ is tentative interval , x^* is exact solution , X is numerical solution , K is iterative number of times , precision $\varepsilon = 1e-5$, entropy factor $p = 300$.

Example 1:

$$M = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}, \quad q = \begin{pmatrix} -4 \\ 3 \\ -4 \\ 2 \end{pmatrix}$$

$x^* = (1, 0, 1, 0)^T$

$$X^{(0)} = \begin{bmatrix} [0,2] \\ [0,2] \\ [0,2] \\ [0,2] \end{bmatrix}, \quad X = \begin{bmatrix} [0.9999987, 0.9999997] \\ [0.0000012, 0.0000042] \\ [0.9999998, 1.0000023] \\ [0.0000043, 0.0000053] \end{bmatrix}, \quad K = 63.$$

Example 2:

$$M = \begin{pmatrix} 3 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 4 & -2 \\ -1 & -1 & -1 & 5 \end{pmatrix}, \quad q = \begin{pmatrix} -2 \\ 3 \\ -4 \\ 5 \end{pmatrix}$$

$x^* = (1, 0, 1, 0)^T$

$$X^{(0)} = \begin{bmatrix} [0,2] \\ [0,2] \\ [0,2] \\ [0,2] \end{bmatrix}, \quad X = \begin{bmatrix} [0.9999877, 0.9999987] \\ [0.0000012, 0.0000024] \\ [0.9999978, 0.9999988] \\ [0.0000014, 0.0000032] \end{bmatrix}, \quad K = 52$$

The examples above and more numerical results indicate that our algorithm works reliably and efficiently.

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