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On Harmonious Colouring of Line Graph of Central Graph of Paths

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Abstract

In this paper, we present some properties of the central graph $C(P_n)$ of a path, and its line graph $L[C(P_n)]$. We mainly have our discussion on the harmonious chromatic number of $C(P_n)$ and the line graph of central graph of P_n, denoted by $L[C(P_n)]$.

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INTRODUCTION

All graphs considered here are undirected. All the notions that are not defined in this paper can be found in [3,14]. The central graph[1,31,33,34] of G, denoted by $C(G)$ is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G. By definition $p_{C(G)} = p+q$, where p and q denotes the number of vertices and edges in G.

The line graph [3,14] $L(G)$ of a graph $G=(V, E)$ is a graph with vertex set $E(G)$ in which two vertices are adjacent if and only if the corresponding edges in G are adjacent.

A harmonious colouring [2,4,6,7,8,9,10,11,12,15,16,18,20,21,22,23,24]is a proper vertex colouring in which every pair of colours appears on at most one pair of adjacent vertices. The harmonious chromatic number $\chi_H(G)$ of a graph *G* is the minimum number of colours needed for any harmonious colouring of *G*; The above definition implies that harmonious colouring is defined for simple graphs rather than multigraphs

1. A BRIEF REVIEW OF HARMONIOUS COLOURING

The first paper on harmonious graph colouring was published in 1982 by Frank Harary and M. J. Plantholt [15]. However, the proper definition of this notion is due to J. E. Hopcroft and M. S. Krishnamoorthy [18] in1983. S.Lee and John Mitchum,[23] published a paper consisting of the upper bound for the harmonious chromatic number of graphs in 1987.

In 1988, Z. Miller and D. Pritikin, worked on harmonious colouring and gave the harmonious chromatic number of graphs, in the Proceedings of 250th Anniversary Conference on Graph Theory (Fort Wayne, Indiana, 1986) (eds. K. S. Bagga et al.), Congressus Numerantium. D.G. Beane, N.L.Biggs and B.J. Wilson, studied the growth rate of harmonious chromatic number in 1989. Again Z. Miller and D. Pritikin [27] published a paper on the topic the harmonious colouring number of a graph in 1991.

In 1991 C. J. H. McDiarmid and Luo Xinhua [26] gave the Upper bounds for harmonious colourings. Zhikang Lu [38] gave a published work on the harmonious chromatic number of a complete binary and trinary tree, in 1993. He also published a paper on Estimates of the harmonious chromatic numbers of some classes of graphs (Chinese), Journal of Systems Science and Mathematical Sciences, 13 (1993).

A combined work by L. R. Casse, C. M. O'Keefe and B. J. Wilson [5] gave us the Minimal harmoniously colourable designs in 1994. In the same year, I. Krasikov and Y. Roditty [21] gave a paper on bounds for the harmonious chromatic number of a graph.

Zhikang Lu,[38] in 1995, published a paper on the harmonious chromatic number of a complete 4-ary tree. Also K. J. Edwards [6] worked and gave results on the harmonious chromatic number of almost all trees. In the next year (1996) he investigated on the harmonious chromatic number of bounded degree trees [7]. John P. Georges [20] published a paper on the harmonious Colourings on collection of graphs in 1995.

In 1996, a paper on the harmonious chromatic number of quasistars, was given by I. Havel and J.M. Laborde Manuscript, Prague and Grenoble, 1996.

In 1997, K. J. Edwards,[8] continued his work on the harmonious chromatic number of bounded degree graphs, and also published papers relating harmonious colouring and achromatic number.

Zhikang Lu [41,42] published a paper on the exact value of the harmonious chromatic number of a complete binary tree in 1997 and trinary tree in 1998.

In 1998, K. J. Edwards [9] published a work emphasizing a new upper bound for the harmonious chromatic number, and in 1999 on the harmonious chromatic number of complete *r*-ary trees.

J. Mitchem and E. Schmeichel, published a paper “The Harmonious Chromatic Number of Deep and Wide Complete *n*-ary Trees”, in The Ninth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications (Kalamazoo, Michigan, 2000) (eds. Y. Alavi, D. Jones, D. R. Lick and Jiuqiang Liu), Electronic Notes in Discrete Mathematics, 11 (2002).

A work on the harmonious chromatic number of $P(\alpha, K_n)$, $P(\alpha, K_{1,n})$ and $P(\alpha, K_{m,n})$, was published by M. F.Mammana [25] in 2003.

D. Campbell and K. J. Edwards [4] again gave a new lower bound for the harmonious chromatic number in 2004.

In 2006, K. Thilagavathi and J. V. Vivin,[30] published a paper “Harmonious colouring of graphs”.

We had a detailed study on the harmonious chromatic number of line graph of central graph of C_n, K_n, $K_{1,n}$, $K_{n,n}$ and its line graphs denoted by $L[C(C_n)]$ $L[C(K_n)]$ $L[C(K_{1,n})]$ and $L[C(K_{n,n})]$ respectively in [31,33,34]

Also we have discussed the harmonious chromatic number of middle graph of central graph of C_n, K_n, $K_{1,n}$ and P_n denoted by $M[C(C_n)], M[C(K_n)], M[C(K_{1,n})]$ and $M[C(P_n)]$ respectively in [1,35].

The problem of harmonious coloring can be applied in such diverse areas as radio navigation, data acquisition and image compression. Unfortunately, the problem is *NP*-hard.

2. Structural Properties of $C(P_n)$

Throughout this paper P_n denotes a path on n edges.

The central graph of P_n, $C(P_n)$ has

(i) $(n+1)$ vertices of degree n.

(ii) n vertices of degree 2.

(iii) The number of vertices in $p_{C(P_n)} = (2n+1)$.

(iv) The number of edges in $q_{C(P_n)} = \frac{n^2+3n}{2}$.

3. STRUCTURAL PROPERTIES OF $L[C(P_n)]$

(i) The number of vertices having degree maximum degree Δ denoted by $n(p_\Delta) = \frac{n(n-1)}{2}$, similarly $n(p_\delta) = 2n$.

(ii) The number of vertices in $L[C(P_n)]$, $p_{L[C(P_n)]} = \frac{n^2 + 3n}{2}$.

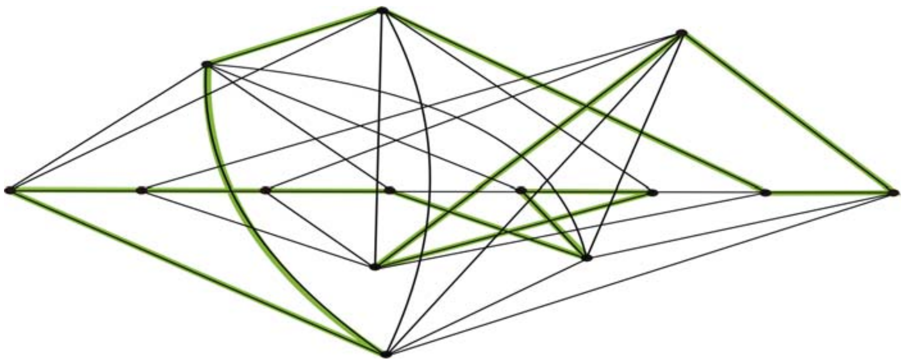
(iii) The number of edges in $L[C(P_n)]$, $q_{L[C(P_n)]} = \frac{n^3 + n}{2}$.

(iv) In $L[C(P_n)]$, minimum degree $\delta = n$ and maximum degree $\Delta = 2(n-1)$. Hence $L[C(P_n)]$ is $(\delta = n, \Delta \equiv 0(\text{mod}\,2))$ – graphs.

(v) $L[C(P_n)]$ is always hamiltonian; if n is even then $L[C(P_n)]$ is both Eulerian and Hamiltonian [17].

Example 3.1

$L[C(P_4)]$



$L[C(P_4)]$ is both Eulerian and Hamiltonian.

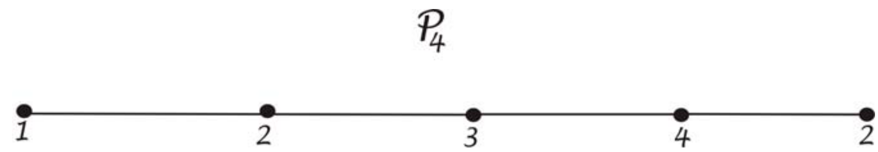
Figure 1

4. HARMONIOUS COLOURING ON *C(P$_n$)* AND $L[C(P_n)]$

Observation 4.1

$\chi_H(P_n) = \{1 + \sqrt{2n+1}\}$.

Example 4.1



$$\chi_H(P_4) = \left\{1 + \sqrt{2(4)+1}\right\} = 4$$

Figure 2

Theorem 4.2

$\chi_H[C(P_n)] = n+3$, *(n >1).*

Proof

Figure 3

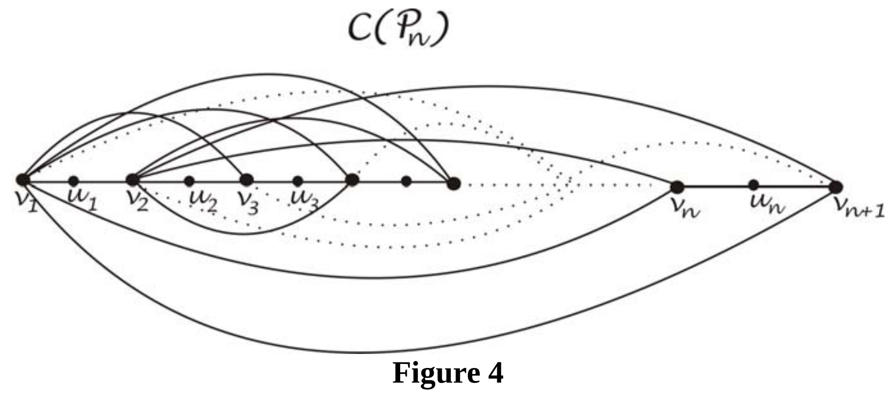
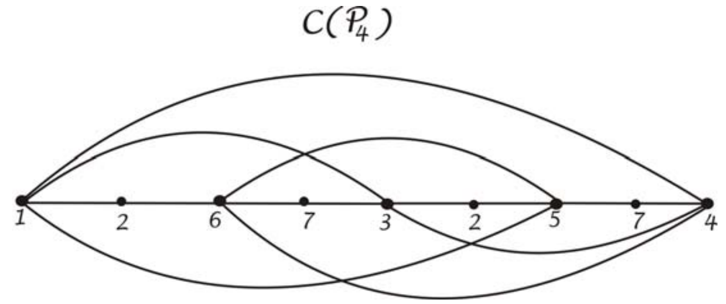


Figure 4

Let $v_1, v_2, \ldots v_{n+1}$ be the vertices of the P_n. Let u_i be the vertex of subdivision of the edge $v_i v_{i+1}$ of P_n. Let $S = \{u_i / 1 \le i \le n\}$. Assign the colour c_i to the vertex v_i $(1 \le i \le n+1)$ assign c_{n+2} to the vertex u_i if i is odd and assign c_{n+3} to the vertex u_i if i is even. Then clearly the above said colouring is a harmonious colouring with minimum number of colours. Therefore $\chi_H[C(P_n)] = n+3$.

Example 4.2



$$\chi_H[C(P_4)] = 4 + 3 = 7$$

Figure 5

Theorem 4.3

$\chi_H\{L[C(P_n)]\} = \dfrac{n^2+3n-2}{2}$, if $n > 2$.

Proof

Let $v_1, v_2, \ldots. v_{n+1}$ be the vertices of the P_n. By the definition of $C(P_n)$, each edge v_iv_{i+1} of P_n is subdivided by the vertex u_i. Clearly the number of edges in $C(P_n)$ is $n + \dfrac{n(n+1)}{2}$.i.e., $q_{C(P_n)} = \dfrac{n^2+3n}{2}$. Each vertex v_i is non adjacent with exactly $(n\text{-}1)$ vertices in P_n. i.e., Each vertex v_i is adjacent with u_i and hence $\deg_{C(P_n)} v_i = n-1+1 = n$ for each $i = 1, 2, \ldots\ldots, n$. v_i and v_{i+1} are non adjacent vertices in $C(P_n)$. Let $E_i = \{e_{i1}, e_{i2}, \ldots\ldots, e_{in}\}$ be the edges in $C(P_n)$ incident with v_i for $i = 1, 2, \ldots\ldots, n$. Clearly the edges $e_{i1}, e_{i2}, \ldots\ldots, e_{in}$ are mutually adjacent for each i. Clearly E_i and E_{i+1} are mutually disjoint. The edges of E_i form a clique K_1 of n vertices in $L[C(P_n)]$ similarly the edges of E_{i+1} form a clique K_2 of n vertices in $L[C(P_n)]$. The edges of K_1 and K_2 are disjoint. Clearly there exist a vertex v of $E\{L[C(P_n)]\} - (K_1 \cup K_2)$ such that v is either adjacent with K_1 or K_2 but not with both. In any harmonious colouring $2n$ colours are assigned to the vertices of K_1 and K_2 and $\binom{n}{2} - 1$ colours are assigned to the vertices of $E\{L[C(P_n)]\} - (K_1 \cup K_2)$. Also it is minimal harmonious colouring.

$$\text{Therefore } \chi_H\{L[C(P_n)]\} = 2n + \binom{n}{2} - 1$$

$$= 2n + \frac{n(n-1)}{2} - 1$$

$$= \frac{4n + n^2 - n - 2}{2}$$

$$= \frac{n^2 + 3n - 2}{2}.$$

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