Applied Mathematical Sciences, Vol. 3, 2009, no. 6, 257 - 265

Exact Solutions for a Class of Boussinesq Equation★

TIAN Bao-Dan♀, QIU Yan-Hong and Chen-Ning

School of Science, Southwest University of Science and Technology
Mianyang, P. R. China, 621010

Abstract. Based on a new projective Riccati equations approach and Maple software, a class of nonlinear evolution equation-Boussinesq equation is discussed in this paper, some new explicit and exact solutions are obtained, such as solitary-wave-like solutions and periodic-like solutions etc. What's more, most of solutions are scarcely seen in the recent research.

Mathematics Subject Classification：35Q53

Keywords: Boussinesq equation, projective Riccati equtions approach, nonlinear evolution equation, explicit and exact solutions

§1 Introduction

Explicit exact solution, especially the explicit solitary wave solutions and periodic wave solutions, which are widely applied in natural science, to the model equations of physical systems are of fundamental importance in physical science and nonlinear science. In the study of systems modeling wave phenomena, one of

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★This work is supported by the Natural Science foundation of Sichuan Education Bureau （under Grant NO.07ZC053）.

♀Corresponding author E-mail: tommycdut@163.com

the important matters of investigation is the traveling wave exact solution, i.e., a moving wave solution with a fixed velocity. It is well known that there are infinite solutions for a nonlinear evolution equation and it is a difficult task to find an exact solution. In recent decades, many powerful approaches have been devised, such as inverse scattering theory[4], Backlund transformation[11], truncated Painléve expansion method[15], Hirota's bilinear method[6], the mixed exponential method[14],Darboux transformation[5], hyperbolic tangent function-series method[3], homogeneous balance method[12], multi-linear variable separation approach[8] and lie group method[10].

Recently, a useful method of finding solitary wave solutions for nonlinear PDE was proposed by Conte and Musette[2]. The key idea of the method is that the traveling wave solutions of nonlinear PDE can be expressed by a polynomial in two variables, which are the components of solution for a projective Riccati equations[1] (PREs for short). This method also belongs to Sub-ODE method. Also, Wang[13] and Lu et.al[9] developed this method and had done some useful work by this approach.

In this paper, we would like to look for some explicit solutions for a class of Boussinesq equation by the projective Riccati equations and Maple software. As we know, the classical nonlinear Boussinesq equation[7] is introduced as follow:

$$u_{tt} - \alpha u_{xx} - \beta (u^2)_{xx} + \gamma u_{xxxx} = 0, \tag{1}$$

The rest of this paper is organized as follows. In Section 2, the main steps of the projective Riccati equations approach is introduced. In Section 3, a serie of solitary-wave-like solutions and periodic-like solutions are derived for the Boussinesq eqution. In Section 4, some conclusion and discussion is provided.

§2 Projective Riccati equations method

In the following we would like to introduce the basic idea of the algorithm of PREs method.For a given nonlinear evolution equation:

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, \cdots) = 0. \tag{2}$$

where the left-hand side of Eq.(2) is a polynomail of $u(x,t)$ and its derivations.And we seek for solutions of the following type:

$$u(x,t)=u(\xi)=\sum_{i=0}^{m}a_i f^i(\xi)+\sum_{j=1}^{m}b_j f^{j-1}(\xi)g(\xi) \tag{3}$$

where a_i, b_j are constants and $\xi=\xi(x,t)$ is an arbitrary function of x,t, $n\in N$ is also a constant which is fixed by considering the banlance between the highest-order nonlinear terms and the highest-order derivations in Eq.(2), while $f(\xi), g(\xi)$ satisfies the following project Riccati equations:

$$\text{(I)}\quad \begin{cases} f'(\xi)=-qf(\xi)g(\xi), \\ g'(\xi)=q[1-g^2(\xi)-rf(\xi)], \\ g^2(\xi)=1-2rf(\xi)+(r^2+\varepsilon)f^2(\xi), \varepsilon=\pm 1 \end{cases} \tag{4}$$

And the Eqs.(4) have the following solution:

$$\begin{cases} f_1(\xi)=\dfrac{a}{b\cosh(q\xi)+c\sinh(q\xi)+ar}; \\ g_1(\xi)=\dfrac{b\sinh(q\xi)+c\cosh(q\xi)}{b\cosh(q\xi)+c\sinh(q\xi)+ar}. \end{cases} \tag{5}$$

Where a,b,c are arbitrary constants satisfies $c^2=a^2+b^2$ when $\varepsilon=1$ and $b^2=a^2+c^2$ when $\varepsilon=-1$;

$$\text{(II)}\quad \begin{cases} f'(\xi)=qf(\xi)g(\xi), \\ g'(\xi)=q[1+g^2(\xi)-rf(\xi)], \\ g^2(\xi)=-1+2rf(\xi)+(1-r^2)f^2(\xi) \end{cases} \tag{6}$$

And the Eqs.(6) have the following solution:

$$\begin{cases} f_2(\xi)=\dfrac{a}{b\cos(q\xi)+c\sin(q\xi)+ar}; \\ g_2(\xi)=\dfrac{b\sin(q\xi)-c\cos(q\xi)}{b\cos(q\xi)+c\sin(q\xi)+ar}. \end{cases} \tag{7}$$

Where a, b, c are arbitrary constants satisfies $a^2=b^2+c^2$. If we substitute (3)、(4)and (3)、(6)into (2), then setting the coefficients of the polynomial $f^i(\xi)g^j(\xi)(i=0,1,2,\cdots; j=0,1)$ to zero yields a set of nonlinear algebra equations (NLAEs).Solving this NLAEs by Maple software we can obtain a series of explicit solutions of Eq.(2).

§3 Exact and explicit solutions to Eq.(1)

In this section we look for the following travelling wave solutions of Eq. (1):

$$u(x,t)=u(\xi),\quad \xi=Lx+Kt+\xi_0$$

Then Eq. (1) is translated into following ODE of $u(\xi)$:

$$(K^2-\alpha L^2)u''-2\beta L^2u'^2-2\beta L^2uu''+\gamma L^4u^{(4)}=0\,. \tag{8}$$

Substitute (3) into (8) and banlance between the highest-order nonlinear terms uu'' (or u'^2) and the highest-order derivations $u^{(4)}$ we obtain

$$m+(m+2)=m+4\Rightarrow m=2\text{, thus,}$$

$$u(\xi)=a_0+a_1f(\xi)+a_2f^2(\xi)+b_1g(\xi)+b_2f(\xi)g(\xi), \tag{9}$$

Case 1. If we substitute formula (4)、(9) into Eq. (1), and set then setting the coefficients of the polynomial $f^i(\xi)g^j(\xi)(i=0,1,,\cdots,6;\, j=0,1)$ to zero, we obtain the following NLAEs by Maple:

$$f^1: q^2K^2a_1+q^4L^4\gamma a_1-\alpha q^2L^2a_1-2\beta q^2L^2a_0a_1-2\beta q^2L^2a_1b_2+2\beta q^2rL^2b_1^2=0,$$

$$f^2: 4q^2K^2a_2+3\alpha q^2rL^2a_1-\beta q^2L^2(10r^2+4\varepsilon)b_1^2-4\alpha q^2L^2a_2+6\beta q^2rL^2a_0a_1+22\beta q^2rL^2b_1b_2$$
$$-15q^4rL^4\gamma a_1-8\beta q^2L^2a_0a_2-3q^2rK^2a_1+16q^4L^4\gamma a_2-4\beta q^2L^2b_2^2-4\beta q^2L^2a_1^2=0,$$

$$f^3: 14\beta q^2rL^2b_2^2-\alpha q^2(r^2+\varepsilon)L^2a_1+25q^4r^2L^4\gamma a_1-65q^4rL^4\gamma a_2-31\beta q^2r^2L^2b_1b_2$$
$$-9\beta q^2L^2a_1a_2+10q^4\varepsilon L^4\gamma a_1-2\beta q^2r^2L^2a_0a_1+5\beta q^2rL^2a_1^2-2\beta q^2\varepsilon L^2a_0a_1+5q^2r(\alpha L^2-K^2)a_2$$
$$+7\beta q^2\varepsilon rL^2b_1^2-11\beta q^2\varepsilon L^2b_1b_2+K^2q^2(r^2+\varepsilon)a_1+7\beta q^2r^3L^2b_1^2+10\beta q^2rL^2a_0a_2=0,$$

$$f^4: 330q^4r^2L^4\gamma a_2-6\alpha q^2(r^2+\varepsilon)L^2a_2-60q^4r(r^2+\varepsilon)L^4\gamma a_1+120q^4\varepsilon L^4\gamma a_2$$
$$+42\beta q^2rL^2a_1a_2-16\beta q^2L^2a_2^2+66\beta q^2r(r^2+\varepsilon)L^2b_1b_2+6K^2q^2(r^2+\varepsilon)a_2-6\beta q^2(r^2+\varepsilon)L^2a_1^2$$
$$-12\beta q^2(r^2+\varepsilon)L^2a_0a_2-64\beta q^2r^2L^2b_2^2-6\beta q^2(r^2+\varepsilon)^2L^2b_1^2-22\beta q^2\varepsilon L^2b_2^2=0,$$

$$f^5: 5\beta q^2r(r^2+\varepsilon)L^2b_2^2-2\beta q^2(r^2+\varepsilon)L^2a_1a_2+2q^4(r^2+\varepsilon)^2L^4\gamma a_1$$
$$-28q^4r(r^2+\varepsilon)L^4\gamma a_2-2\beta q^2L^2(r^2+\varepsilon)^2b_1b_2=0,$$

$$f^6: 6q^4L^4(r^2+\varepsilon)^2\gamma a_2-\beta q^2L^2(r^2+\varepsilon)^2b_2^2-\beta q^2L^2(r^2+\varepsilon)a_2^2=0,$$

$$f^5g: 3q^4L^4(r^2+\varepsilon)^2\gamma b_2-\beta q^2L^2(r^2+\varepsilon)a_2b_2=0,$$

$$f^4g: 3q^4L^4(r^2+\varepsilon)^2\gamma b_1-30q^4L^4r(r^2+\varepsilon)\gamma b_2-3\beta q^2L^2(r^2+\varepsilon)a_2b_1$$
$$-3\beta q^2L^2(r^2+\varepsilon)a_1b_2+7\beta q^2L^2ra_2b_2=0,$$

$$f^3g: 10q^4L^4\varepsilon\gamma b_2-3\beta q^2L^2a_2b_2-\alpha q^2L^2(r^2+\varepsilon)b_2+5\beta q^2L^2r(a_1b_2+a_2b_1)-6q^4L^4r(r^2+\varepsilon)\gamma b_1$$
$$+q^2K^2(r^2+\varepsilon)b_2+25q^4L^4r^2\gamma b_2-2\beta q^2L^2(r^2+\varepsilon)a_1b_1-2\beta q^2L^2(r^2+\varepsilon)a_0b_2=0,$$

$$f^2g: 3\alpha q^2L^2rb_2-4\beta q^2L^2(a_2b_1+a_1b_2)-15q^4L^4r\gamma b_2+q^2K^2(r^2+\varepsilon)b_1+6\beta q^2L^2r(a_1b_1+a_0b_2)$$
$$-\alpha q^2L^2(r^2+\varepsilon)b_1-3q^2K^2rb_2-2\beta q^2L^2(r^2+\varepsilon)a_0b_1+4q^4L^4\varepsilon\gamma b_1+7q^4L^4r^2\gamma b_1=0,$$
$$fg: q^2K^2(b_2-3rb_1)-2\beta q^2L^2(a_1b_1+a_0b_2)+2\beta q^2L^2ra_0b_1+(q^4L^4\gamma-\alpha q^2L^2)(b_2-rb_1)=0.$$

Solving the above NLAEs by Maple software,one can get the following four sets of solutions,

$$a_0=\frac{K^2-\alpha L^2+\gamma q^2L^4}{2\beta L^2}, a_1=-\frac{3\gamma q^2L^2r}{\beta}, a_2=\frac{3\gamma q^2L^2(r^2+\varepsilon)}{\beta}, b_1=0, b_2=\frac{3\gamma q^2L^2\sqrt{r^2+\varepsilon}}{\beta}, \text{ when } r\neq 0;$$

$$a_0=\frac{K^2-\alpha L^2+4\gamma q^2L^4}{2\beta L^2},\ a_1=0,\ a_2=\frac{6\gamma q^2L^2\varepsilon}{\beta},\ b_1=b_2=0, \text{ when } r=0;$$

$$a_0=\frac{K^2-\alpha L^2+\gamma q^2L^4}{2\beta L^2}, a_1=0, a_2=\frac{3\gamma q^2L^2\varepsilon}{\beta}, b_1=0, b_2=\pm\frac{3\gamma q^2L^2\sqrt{\varepsilon}}{\beta}, \text{ when } r=0;$$

Thus at this time, we can obtain the following solitary-wave-like solutions of the Eq.(1),

$$u_1(x,t)=\frac{K^2-\alpha L^2+\gamma q^2L^4}{2\beta L^2}-\frac{3a\gamma q^2L^2r}{\beta[b\cosh(q\xi)+c\sinh(q\xi)+ar]}+\frac{3a^2\gamma q^2L^2(r^2+\varepsilon)}{\beta[b\cosh(q\xi)+c\sinh(q\xi)+ar]^2}$$
$$+\frac{3a\gamma q^2L^2\sqrt{r^2+\varepsilon}[b\sinh(q\xi)+c\cosh(q\xi)]}{\beta[b\cosh(q\xi)+c\sinh(q\xi)+ar]^2}; \quad (10)$$

and $$u_2(x,t)=\frac{K^2-\alpha L^2+4\gamma q^2L^4}{2\beta L^2}+\frac{6a^2\gamma q^2L^2\varepsilon}{\beta[b\cosh(q\xi)+c\sinh(q\xi)]^2}; \quad (11)$$

$$u_{3,4}(x,t)=\frac{K^2-\alpha L^2+\gamma q^2L^4}{2\beta L^2}-\frac{3a^2\gamma q^2L^2\varepsilon}{\beta[b\cosh(q\xi)+c\sinh(q\xi)]^2}\pm\frac{3a\gamma q^2L^2\sqrt{\varepsilon}[b\sinh(q\xi)+c\cosh(q\xi)]}{\beta[b\cosh(q\xi)+c\sinh(q\xi)]^2}. \quad (12)$$

Where a, b, c are arbitrary constants satisfies $c^2=a^2+b^2$ when $\varepsilon=1$ and $b^2=a^2+c^2$ when $\varepsilon=-1$;

Case 2. If we substitute formula (6)、(9) into Eq. (1), and set then setting the coefficients of the polynomial $f^i(\xi)g^j(\xi)(i=0,1,,\cdots,6; j=0,1)$ to zero, we also obtain the following NLAEs by Maple,

$$f^1: (\alpha L^2-K^2)q^2a_1+2\beta q^2L^2a_0a_1+2\beta q^2rL^2b_1^2-2\beta q^2L^2b_1b_2+q^4L^4\gamma a_1=0,$$

$$f^2: 4q^2(\alpha L^2-K^2)a_2-3q^2r(\alpha L^2-K^2)a_1+8\beta q^2L^2a_0a_2-15q^4rL^4\gamma a_1+16q^4L^4\gamma a_2$$
$$-\beta q^2L^2(10r^2-4)b_1^2+4\beta q^2L^2(a_1^2-b_2^2)-6\beta q^2rL^2a_0a_1+22\beta q^2rL^2b_1b_2=0,$$

$$f^3: 7\beta q^2L^2r(r^2-1)b_1^2+\alpha q^2L^2(r^2-1)a_1-K^2q^2(r^2-1)a_1+5q^2r(K^2-\alpha L^2)a_2$$
$$+9\beta q^2L^2a_1a_2+2\beta q^2L^2(r^2-1)a_0a_1+14\beta q^2rL^2b_2^2+25q^4r^2L^4\gamma a_1-5\beta q^2rL^2a_1^2$$
$$+\beta q^2L^2(11-31r^2)b_1b_2-65q^4rL^4\gamma a_2-10q^4L^4\gamma a_1-10\beta q^2rL^2a_0a_2=0,$$

$$f^4: 3\alpha q^2L^2(r^2-1)a_2+\beta q^2L^2(11-32r^2)b_2^2-30q^4\gamma L^4r(r^2-1)a_1+6\beta q^2L^2(r^2-1)a_0a_2$$
$$-21\beta q^2rL^2a_1a_2+3\beta q^2L^2(r^2-1)a_1^2+q^4L^4\gamma(115r^2-60)a_2-3\beta q^2L^2(r^2-1)^2b_1^2$$
$$+8\beta q^2L^2a_2^2-3K^2q^2(r^2-1)a_2+33\beta q^2r(r^2-1)L^2b_1b_2=0,$$
$$f^5: 5\beta q^2L^2r(r^2-1)b_2^2-3\beta q^2L^2ra_2^2+2\gamma q^4L^4(r^2-1)^2a_1-28q^4r(r^2-1)L^4\gamma a_2$$
$$+2\beta q^2L^2(r^2-1)a_1a_2-2\beta q^2L^2(r^2-1)^2b_1b_2=0,$$
$$f^6:\ 6\gamma q^4L^4(r^2-1)^2a_2+\beta q^2L^2(r^2-1)a_2^2-\beta q^2L^2(r^2-1)^2b_2^2=0,$$
$$f^5g: \beta q^2L^2(r^2-1)a_2b_2+3\gamma q^4L^4(r^2-1)^2b_2=0,$$
$$f^4g: 3\gamma q^4L^4(r^2-1)^2b_1+3\beta q^2L^2(r^2-1)(a_1b_2+a_2b_1)-30\gamma q^4L^4r(r^2-1)b_2-7\beta q^2L^2ra_2b_2=0,$$
$$f^3g: \alpha q^2L^2(r^2-1)b_2-6\gamma q^4L^4r(r^2-1)b_1-K^2q^2(r^2-1)b_2+2\beta q^2L^2(r^2-1)(a_1b_1+a_0b_2)$$
$$+5\gamma q^4L^4(5r^2-2)b_2+3\beta q^2L^2a_2b_2-5\beta q^2L^2r(a_2b_1+a_1b_2)=0,$$
$$f^2g: 2\beta q^2L^2(r^2-1)a_0b_1-15\gamma q^4L^4rb_2-6\beta q^2L^2r(a_1b_1+a_0b_2)+\gamma q^4L^4(7r^2-4)b_1$$
$$-K^2q^2(r^2-1)b_1+4\beta q^2L^2(a_1b_2+a_2b_1)+3q^2r(K^2-\alpha L^2)b_2+\alpha q^2L^2(r^2-1)b_1=0,$$
$$fg:\ q^2(\alpha L^2-K^2)b_2+2\beta q^2L^2(a_1b_1+a_0b_2-ra_0b_1)+\gamma q^4L^4(b_2-rb_1)-q^2r(\alpha L^2-K^2)b_1=0,$$

Solving the above NLAEs by Maple software,one can also get the following five sets of solutions,

$$a_0=\frac{K^2-\alpha L^2-\gamma q^2L^4}{2\beta L^2},\ a_1=\frac{3\gamma q^2L^2r}{\beta},\ a_2=\frac{3\gamma q^2L^2(1-r^2)}{\beta},\ b_1=0,\ b_2=\pm\frac{3\gamma q^2L^2\sqrt{1-r^2}}{\beta},\text{when } r\neq 0;$$
$$a_0=\frac{K^2-\alpha L^2-\gamma q^2L^4}{2\beta L^2},\quad a_1=0,\quad a_2=\frac{3\gamma q^2L^2}{\beta},\quad b_1=0,\quad b_2=\pm\frac{3\gamma q^2L^2}{\beta},\ \text{when } r=0;$$
$$a_0=\frac{K^2-\alpha L^2-4\gamma q^2L^4}{2\beta L^2},\quad a_1=0,\quad a_2=\frac{6\gamma q^2L^2}{\beta},\quad b_1=b_2=0,\ \text{when } r=0;$$

And we can obtain the following periodic-like solutions of the Eq.(1),

$$u_{5,6}(x,t)=\frac{K^2-\alpha L^2-\gamma q^2L^4}{2\beta L^2}+\frac{3\gamma q^2L^2ra}{\beta[b\cos(q\xi)+c\sin(q\xi)+ar]}+\frac{3\gamma q^2L^2(1-r^2)a^2}{\beta[b\cos(q\xi)+c\sin(q\xi)+ar]^2}$$
$$\pm\frac{3a\gamma q^2L^2\sqrt{1-r^2}[b\sin(q\xi)-c\cos(q\xi)]}{\beta[b\cos(q\xi)+c\sin(q\xi)+ar]^2};\tag{13}$$

$$u_{7,8}(x,t)=\frac{K^2-\alpha L^2-\gamma q^2L^4}{2\beta L^2}+\frac{3\gamma q^2L^2a^2}{\beta[b\cos(q\xi)+c\sin(q\xi)]^2}\pm\frac{3\gamma q^2L^2a[b\sin(q\xi)-c\cos(q\xi)]}{\beta[b\cos(q\xi)+c\sin(q\xi)]^2};\tag{14}$$

$$u_9(x,t)=\frac{K^2-\alpha L^2-4\gamma q^2L^4}{2\beta L^2}+\frac{6\gamma q^2L^2a^2}{\beta[b\cos(q\xi)+c\sin(q\xi)]^2};\tag{15}$$

where $a^2=b^2+c^2$ in the formula (13)—(15).

§4 Conclusions and discussion

In this paper, the authors are successfully to get a series of new explicit solutions for a class of Boussinesq equation, such as solitary-wave-like solutions and periodic-like solutions etc., and most of them are scarcely seen in the recent research. What' more, the solutions by this approach is a generalization of the traditional Projective Riccati equations.

For example, if we take $a=b=q=1, c=0$, we can obtain the following type of explicit solutions:

$$u_1^*(x,t)=\frac{K^2-\alpha L^2+\gamma q^2L^4}{2\beta L^2}-\frac{3\gamma L^2 r}{\beta[\cosh(\xi)+r]}+\frac{3\gamma L^2(r^2-1)}{\beta[\cosh(\xi)+r]^2}+\frac{3\gamma L^2\sqrt{r^2-1}}{\beta[\coth(\xi)+r\,\mathrm{csch}(\xi)]^2};$$

$$u_2^*(x,t)=\frac{K^2-\alpha L^2+4\gamma q^2L^4}{2\beta L^2}-\frac{6\gamma L^2}{\beta}\sec h^2(\xi);$$

$$u_{3,4}^*(x,t)=\frac{K^2-\alpha L^2+\gamma q^2L^4}{2\beta L^2}+\frac{3\gamma L^2}{\beta}[\mathrm{sech}^2(\xi)\pm i\,\mathrm{sech}(\xi)\tanh(\xi)];$$

$$u_{5,6}^*(x,t)=\frac{K^2-\alpha L^2-\gamma q^2L^4}{2\beta L^2}+\frac{3\gamma L^2(1-r^2)}{\beta[b\cos(\xi)+r]^2}+\frac{3\gamma L^2 r}{\beta[\cos(\xi)+r]}\pm\frac{3\gamma L^2\sqrt{1-r^2}\sin(\xi)}{\beta[\cos(\xi)+r]^2};$$

$$u_{7,8}^*(x,t)=\frac{K^2-\alpha L^2-\gamma q^2L^4}{2\beta L^2}+\frac{3\gamma L^2}{\beta}[\mathrm{sech}^2(\xi)\pm\mathrm{sech}(\xi)\tanh(\xi)];$$

$$u_9^*(x,t)=\frac{K^2-\alpha L^2-4\gamma q^2L^4}{2\beta L^2}+\frac{6\gamma L^2}{\beta}\sec h^2(\xi).$$

In fact, this method is also applicable to other nonlinear evolution equations in physics. Moreover, we are also able to discuss the explicit solutions for the nonlinear evolution equations with variant coefficients, and this work lives for the future research.

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Received: June 10, 2008