A Note on Grüss Type Inequality

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Abstract. In this short note, we establish a new form of the inequality of Grüss type for functions whose first and second derivatives are absolutely continuous and third derivative is bounded both above and below almost everywhere.

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1. Introduction

Let \( f \) and \( g \) be two bounded functions defined on \([a, b]\) with \( \gamma_1 \leq f(x) \leq \Gamma_1 \) and \( \gamma_2 \leq g(x) \leq \Gamma_2 \), where \( \gamma_1, \gamma_2, \Gamma_1, \Gamma_2 \) are four constants. Then the classic Grüss inequality reads as follows:

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \frac{1}{b-a} \int_a^b g(x)dx \leq \frac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2).
\]

In the years thereafter, numerous generalizations, extensions and variants of Grüss inequality have appeared in the literature (see [1, 2, 3, 4, 5, 6, 7, 8, 9]). The purpose of the present note is to establish a new form of the inequality of Grüss type for functions whose first and second derivatives are absolutely continuous and third derivative is bounded both above and below almost everywhere.

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2. Gr"{u}ss inequality

In this section, we shall obtain the following main result.

**Theorem 2.1.** Let \( f : [a, b] \rightarrow (-\infty, \infty) \) be a function such that the derivative \( f', f'' \) is absolutely continuous on \([a, b]\). Assume that there exist constants \( \gamma, \Gamma \in (-\infty, \infty) \) such that \( \gamma \leq f'''(x) \leq \Gamma \) a.e. on \([a, b]\). Then we have

\[
\left| \left( a^2 + ba + b^2 \right)(bf''(a) - af''(b)) - 3(b^2 f'(b) - a^2 f'(a)) \\
+ 6(bf(b) - af(a)) - \int_a^b f(x)dx \right|
\leq (\Gamma - \gamma) \frac{b^4 + 3C^{4/3} - 4bC}{4},
\]

where

\[
C = \frac{(b + a)(b^2 + a^2)}{4}.
\]

**Proof.** Firstly, it is easy to check

\[
(a^2 + ba + b^2)(bf''(a) - af''(b)) - 3(b^2 f'(b) - a^2 f'(a)) \\
+ 6(bf(b) - af(a)) - \int_a^b f(x)dx
\]

\[
= b^2 f''(b) - a^2 f''(a) - 3(b^2 f'(b) - a^2 f'(a)) + 6(bf(b) - af(a)) \\
- (b + a)(b^2 + a^2)[f''(b) - f''(a)] - \int_a^b f(x)dx
\]

\[
= \int_a^b \left\{ x^3 - \frac{1}{b - a} \int_a^b x^3dx \right\} f'''(x)dx.
\]

Let

\[
A = \left\{ x \in [a, b] : x^3 \geq \frac{1}{b - a} \int_a^b x^3dx \right\};
\]

\[
A^c = \left\{ x \in [a, b] : x^3 < \frac{1}{b - a} \int_a^b x^3dx \right\}.
\]

Then we have

\[
\int_{A^c} \left\{ x^3 - \frac{1}{b - a} \int_a^b x^3dx \right\} f'''(x)dx
\]

\[
\leq \Gamma \int_{A^c} \left\{ x^3 - \frac{1}{b - a} \int_a^b x^3dx \right\} dx + \gamma \int_{A^c} \left\{ x^3 - \frac{1}{b - a} \int_a^b x^3dx \right\} dx
\]
and
\[
\int_{a}^{b} \left\{ x^3 - \frac{1}{b-a} \int_{a}^{b} x^3 \, dx \right\} f'''(x) \, dx \\
\geq \gamma \int_{A} \left\{ x^3 - \frac{1}{b-a} \int_{a}^{b} x^3 \, dx \right\} \, dx + \Gamma \int_{A'} \left\{ x^3 - \frac{1}{b-a} \int_{a}^{b} x^3 \, dx \right\} \, dx.
\]
Since
\[
\int_{A} \left\{ x^3 - \frac{1}{b-a} \int_{a}^{b} x^3 \, dx \right\} \, dx = - \int_{A'} \left\{ x^3 - \frac{1}{b-a} \int_{a}^{b} x^3 \, dx \right\} \, dx,
\]
it follows that
\[
(2.1) \quad \left| \int_{a}^{b} \left\{ x^3 - \frac{1}{b-a} \int_{a}^{b} x^3 \, dx \right\} f'''(x) \, dx \right|
\leq (\Gamma - \gamma) \int_{A} \left\{ x^3 - \frac{1}{b-a} \int_{a}^{b} x^3 \, dx \right\} \, dx
\leq (\gamma - \Gamma) \int_{A'} \left\{ x^3 - \frac{1}{b-a} \int_{a}^{b} x^3 \, dx \right\} \, dx.
\]
Therefore, it is enough to discuss the following integral,
\[
(2.2) \quad \int_{A} \left\{ x^3 - \frac{1}{b-a} \int_{a}^{b} x^3 \, dx \right\} \, dx.
\]
From the definition of the set \( A \), it follows that
\[
A = \left\{ x \in [a, b]; \sqrt[3]{\frac{(b+a)(b^2+a^2)}{4}} \leq x \leq b \right\},
\]
and we can claim that
\[
(2.3) \quad a \leq \sqrt[3]{\frac{(b+a)(b^2+a^2)}{4}} \leq b, \quad \forall \ a < b.
\]
In fact, we can assume \( b = ka \), where \( k \) is chosen from \( R \) based on \( a \). If \( a \geq 0 \) which implies \( b > 0 \), then \( k > 1 \) and the inequality (2.3) is equivalent to
\[
1 \leq \frac{(k+1)(k^2+1)}{4} \leq k^3
\]
which is obvious. Similarly if \( a < 0, b \leq 0 \), then \( 0 \leq k \leq 1 \) and the inequality (2.3) is equivalent to
\[
(2.4) \quad 1 \geq \sqrt[3]{\frac{(k+1)(k^2+1)}{4}} \geq k,
\]
if \( a < 0, b \geq 0 \), then \( k \leq 0 \) and the inequality (2.3) is equivalent also to
\[
(2.5) \quad 1 \geq \sqrt[3]{\frac{(k+1)(k^2+1)}{4}} \geq k,
\]
It is easy to see (2.4) and (2.5) hold correspondingly. Hence the integral (2.2) can be obtained,

\[
\int_A \left\{ x^3 - \frac{1}{b-a} \int_a^b x^3 \, dx \right\} \, dx
\]

\[
= \int_a^b \frac{x^3 - \frac{1}{b-a} \int_a^b x^3 \, dx}{\sqrt{\frac{(b-a)(b^2+a^2)}{4}}} \, dx
\]

\[
= b^4 - \frac{(b+a)(b^2+a^2)}{4} \left( \frac{4}{3} \right) - \frac{(a+b)(a^2+b^2)}{4} \left[ b - \left( \frac{(b+a)(b^2+a^2)}{4} \right)^{1/3} \right]
\]

\[
= \frac{b^4 + 3 \left( \frac{(b+a)(b^2+a^2)}{4} \right)^{4/3} - 4b \frac{(b+a)(b^2+a^2)}{4}}{4}.
\]

The desired result can be obtained. □

REFERENCES


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