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On the Degree of Approximation of Functions from Lipschitz Classes

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Abstract

In this paper we obtained the degree of approximation of functions belonging to Lipschitz classes, with double Fourier series, using Euler means.

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1. Introduction and Preliminaries

A lot of authors investigated the degree of approximation of functions from Lipschitz classes $Lip\alpha$, $0 < \alpha \leq 1$, with Fourier series using different means. G. Alexits [1] has used the (C, δ) means of Fourier series. In [2] Hölland, Sahney and Tzimbalario have used the Nörlund (N, p_n) means of Fourier series, then H. H. Khan and A. Wafi [3] have given the answer to an open problem imposed by Hölland, Sahney and Tzimbalario in [2] using matrix means of Fourier series.

Later in [4] H. H. Khan and G. Ram have considered the problem of determining the degree of approximation using Euler means $(E, q), q > 0$ of Fourier series.

Let $f(x, y)$ be a 2π-periodic function by each variable and integrable in the sense of Lebesgue. Suppose that double series

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}(x, y),$$

is Fourier series of function $f(x, y)$, and $S_{kl}(x, y)$ are its partial sums, where

$$A_{00}(x, y) = \frac{1}{4} a_{00},$$

$$A_{m0}(x,y)=\frac{1}{2}(a_{m0}\cos mx+b_{m0}\sin mx),$$
$$A_{0n}(x,y)=\frac{1}{2}(a_{0n}\cos nx+b_{0n}\sin nx),$$

$$\begin{aligned}A_{mn}(x,y) &= a_{mn}\cos mx\cos ny+b_{mn}\sin mx\cos ny\\ &+ c_{mn}\cos mx\sin ny+d_{mn}\sin mx\sin ny.\end{aligned}$$

Is not difficult to see that

$$S_{kl}(x,y)=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x+t_1,y+t_2)D_k(t_1)D_l(t_2)dt_1dt_2, \tag{1}$$

where $D_r(t)=\frac{\sin\left(r+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$ is the Dirichlet kernel.

A 2π periodic function $f(x,y)$ in each variable x and y is said to belong to the class $Lip(\varphi(t_1,t_2);p), p>1$, (see [5], [6]), if

$$|f(x+t_1,y+t_2)-f(x,y)|\le M\frac{\varphi(t_1,t_2)}{(t_1\cdot t_2)^{1/p}}, 0<t_i\le\pi, i=1,2,$$

where $\varphi(t_1,t_2)$ is a positive increasing function of the variables t_1,t_2 and M is a positive constant indenpendent of x,y,t_1 and t_2.

In [5] author Huzoor H. Khan defined the operator $L_{ns-s,ms-s}(f;x,y)$, (n, m, s are positive integers) by this equality

$$\begin{aligned}&L_{ns-s,ms-s}(f;x,y)=\\ =&\frac{1}{A_{ns-s,ms-s}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x+u,y+v)\left(\frac{\sin\frac{1}{2}nu}{\sin\frac{1}{2}u}\right)^{2s}\left(\frac{\sin\frac{1}{2}mv}{\sin\frac{1}{2}v}\right)^{2s}dudv,\end{aligned}$$

where

$$A_{ns-s,ms-s}=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(\frac{\sin\frac{1}{2}nu}{\sin\frac{1}{2}u}\right)^{2s}\left(\frac{\sin\frac{1}{2}mv}{\sin\frac{1}{2}v}\right)^{2s}dudv.$$

Before we formulate his result we make mention that he put

$$\begin{aligned}\phi(u,v) &= f(x+u,y+v)+f(x-u,y+v)\\ &+ f(x+u,y-v)+f(x-u,y-v)-4f(x,y).\end{aligned}$$

His result states as follows

Theorem H. Let $f(x,y)$ be a periodic function of peridod 2π with respect to each variable x and y belonging $Lip(\varphi(u,v);p)$, for $p>1$ class, then

$$\begin{aligned}E_n^*(f) &= \min_{L_n}|L_{ns-s,ms-s}(f;x,y)-f(x,y)|\\ &= O\left\{\varphi\left(\frac{1}{n},\frac{1}{m}\right)\left(\frac{1}{n},\frac{1}{m}\right)^{-\frac{1}{p}}\right\}, (n,m=1,2,...;s\ge 3)\end{aligned}$$

provided

$$\left\{\int_0^{\frac{\pi}{2n}}\int_0^{\frac{\pi}{2m}}\left(\frac{(u,v)|\phi(2u,2v)|}{\varphi(2u,2v)}\right)^p dudv\right\}^{\frac{1}{p}} = O\left(\frac{1}{n},\frac{1}{m}\right)$$

and

$$\left\{\int_{\frac{\pi}{2n}}^{\pi}\int_{\frac{\pi}{2m}}^{\pi}\left(\frac{(u,v)^{-\delta}|\phi(2u,2v)|}{\varphi(2u,2v)}\right)^p dudv\right\}^{\frac{1}{p}} = O\left(n^{\delta},m^{\delta}\right)$$

where δ is an arbitrary number such that $q(1-\delta)-1>0$ and $p+q=pq$.

Later in [6] authors H. H. Khan and G. Ram investigated the degree of approximation of functions belonging to class Lipschitz using the Gauss Weierstrass integral of double Fourier series.

They defined the Gauss Weierstrass integral of $f(x,y)$ by

$$W_{mn}(x,y)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{\exp((k^2\xi/4)+(l^2\eta/4))}A_{k,l}(x,y),$$

where $\xi\to 0$, $\eta\to 0$.

Their result is this

Theorem HG. Let $f(x,y)$ be a continuous function of period 2π with respect to each variable x and y belonging to $Lip(\psi(u,v);p), p>1$ class, then

$$|W_{mn}(x,y;\xi,\eta)-f(x,y)| = O\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{(1/p)-(1/2)}}\right)$$

provided

$$\left[\int_0^{\xi}\int_0^{\eta}\left(\frac{\psi(t,s)}{(t,s)^{1/p}}\right)^p dtds\right]^{1/p} = O\left(\psi(\xi,\eta)\right)$$

and

$$\left[\int_0^{\xi}\int_{\eta}^{\pi}\left(\frac{\psi(t,s)}{(t,s)^{2+(1/p)}}\right)^p dtds\right]^{1/p} = O\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^2}\right).$$

Let $\{S_{mn}\}$ be the sequence of partial sums of the given double series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}u_{mn}$. Then for $p,q>0$ real numbers the Euler means (E,p,q) of the sequence $\{S_{mn}\}$ we define by

$$E_{mn}^{pq}=(1+p)^{-m}(1+q)^{-n}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}p^{m-k}q^{n-l}S_{kl}.$$

The series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}u_{mn}$ is said to be Euler (E,p,q) summable to number S provided that the sequence $\{E_{mn}^{pq}\}$ converges to S as $m,n\to\infty$.

In [7] M. Topolewska introduced the Euler mean $E^{pq}_{mn}[f], (p,q>0)$ of the partial sums of double Fourier series of a function$f$ defined in the square $Q=[-\pi,\pi]^2=[-\pi,\pi]\times[-\pi,\pi]$ and 2π-periodic in each variable by the formula

$$E^{pq}_{mn}[f](x,y)=(1+p)^{-m}(1+q)^{-n}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}p^{m-k}q^{n-l}S_{kl}[f](x,y).$$

The object of this paper is to determine the degree of approximation for the functions belonging to the class $Lip(\varphi(t_1,t_2);p), p>1$, by means of Euler $E^{pq}_{mn}[f](x,y)$ of the double Fourier series of $f(x,y)$. In fact our results are two dimensional respectively $n-$dimensional cases of result proved in [4].

We shall use notations

$$\begin{aligned}
\tau(t_1,t_2) &= \frac{1}{4}\{f(x+t_1,y+t_2)+f(x-t_1,y+t_2)+\\
&+ f(x+t_1,y-t_2)+f(x-t_1,y-t_2)-4f(x,y)\},\\
S(u) &= \sum_{i=0}^{r}\binom{r}{i}a^{r-i}\sin\left(i+\frac{1}{2}\right)u,\\
R(u) &= \sin\left\{\frac{u}{2}+r\tan^{-1}\left(\frac{\sin u}{a+\cos u}\right)\right\},\ a>0.
\end{aligned}$$

We need following lemma proved in [4]:

Lemma 1. If $0<u\le\pi$, $s\in N$ then

$$(1+r)^{-s}(1+r^2+2r\cos u)^{s/2}=O(1)e^{-2ru^2s/\{\pi(1+r)\}^2},\ r>0.$$

2. Main results

2.1 Two dimensional case

We shall prove this theorem as main result

Theorem 1. Let $f(x,y)$ be 2π-periodic function by each variable and belongs to the class $Lip(\varphi(t_1,t_2);p)$, for $p>1$. If

$$\int_0^{1/\sqrt{m}}\int_0^{1/\sqrt{n}}\frac{\varphi^p(t_1,t_2)}{t_1t_2}dt_1dt_2=O\left\{\varphi^p\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\right\},$$

$$\int_{1/\sqrt{m}}^{\pi}\int_0^{1/\sqrt{n}}\frac{\varphi^p(t_1,t_2)}{(t_1t_2)^{1+p}\,t_1^p}dt_1dt_2=O\left\{\varphi\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)m\right\}^p,$$

$$\int_0^{1/\sqrt{m}}\int_{1/\sqrt{n}}^{\pi}\frac{\varphi^p(t_1,t_2)}{(t_1t_2)^{1+p}\,t_2^p}dt_1dt_2=O\left\{\varphi\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)n\right\}^p,$$

$$\int_{1/\sqrt{m}}^{\pi}\int_{1/\sqrt{n}}^{\pi}\frac{\varphi^p(t_1,t_2)}{(t_1t_2)^{1+2p}}dt_1dt_2 = O\left\{\varphi\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)mn\right\}^p,$$

then

$$\max_{0\le x,y\le 2\pi}|E_{mn}^{pq}[f](x,y)-f(x,y)| = O\left\{\varphi\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)(mn)^{1/2p}\right\}.$$

Proof. From (1) we can show easily that

$$S_{kl}(x,y)-f(x,y)=\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\tau(t_1,t_2)\frac{\sin\left(k+\frac{1}{2}\right)t_1}{\sin\frac{t_1}{2}}\frac{\sin\left(l+\frac{1}{2}\right)t_2}{\sin\frac{t_2}{2}}dt_1dt_2. \quad (2)$$

From (2) we have

$$(1+p)^{-m}(1+q)^{-n}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}p^{m-k}q^{n-l}\{S_{kl}(x,y)-f(x,y)\}=$$

$$=\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\frac{\tau(t_1,t_2)}{\sin\frac{t_1}{2}\cdot\sin\frac{t_2}{2}}\cdot\frac{S(t_1)}{(1+p)^m}\cdot\frac{S(t_2)}{(1+q)^n}dt_1dt_2.$$

Therefore $(S(t_1,t_2)=S(t_1)\cdot S(t_2))$

$$\begin{aligned}
|E_{mn}^{pq}[f](x,y) &- f(x,y)| = \\
&= O\left(\frac{1}{\pi^2}\right)\int_0^{\pi}\int_0^{\pi}\left|\frac{\tau(t_1,t_2)}{\sin\frac{t_1}{2}\cdot\sin\frac{t_2}{2}}\cdot\frac{S(t_1,t_2)}{(1+p)^m(1+q)^n}\right|dt_1dt_2 \\
&= O\left(\frac{1}{\pi^2}\right)\left\{\int_0^{1/\sqrt{m}}\int_0^{1/\sqrt{n}}+\int_0^{1/\sqrt{m}}\int_{1/\sqrt{n}}^{\pi}+\int_{1/\sqrt{m}}^{\pi}\int_0^{1/\sqrt{n}}+\right. \\
&+ \left.\int_{1/\sqrt{m}}^{\pi}\int_{1/\sqrt{n}}^{\pi}\right\} = O\left(\frac{1}{\pi^2}\right)\sum_{i=1}^{4}I_i(x,y). \quad (3)
\end{aligned}$$

By Hölder's inequality we have :

$$\begin{aligned}
I_1(x,y) &= \int_0^{1/\sqrt{m}}\int_0^{1/\sqrt{n}}\frac{|\tau(t_1,t_2)|}{\sin\frac{t_1}{2}\cdot\sin\frac{t_2}{2}}\cdot\frac{|S(t_1,t_2)|}{(1+p)^m(1+q)^n}dt_1dt_2 \\
&= O(1)\left[\int_0^{1/\sqrt{m}}\int_0^{1/\sqrt{n}}|\tau(t_1,t_2)|^p dt_1dt_2\right]^{1/p}\times \\
&\times \left[\int_0^{1/\sqrt{m}}\int_0^{1/\sqrt{n}}\left|\frac{S(t_1,t_2)}{(1+p)^m(1+q)^n\sin\frac{t_1}{2}\cdot\sin\frac{t_2}{2}}\right|^{p'}dt_1dt_2\right]^{1/p'},
\end{aligned}$$

where $p+p'=pp'$.

According to lemma 1 is valid this estimate

$$\frac{|S(t_1,t_2)|}{(1+p)^m(1+q)^n}=O(1)e^{-2pt_1^2m/\{\pi(1+p)\}^2-2qt_2^2n/\{\pi(1+q)\}^2},$$

and the fact that $Lip(\varphi(t_1,t_2);p)$ we have

$$\begin{aligned}
I_1(x,y) &= O(1)\left[\int_0^{1/\sqrt{m}}\int_0^{1/\sqrt{n}}\frac{\varphi^p(t_1,t_2)}{t_1t_2}dt_1dt_2\right]^{1/p}\times\\
&\times \left\{\int_0^{1/\sqrt{m}}\left(\frac{e^{-2pt_1^2m/\{\pi(1+p)\}^2}}{|\sin\frac{t_1}{2}|}\right)^{p'}dt_1\int_0^{1/\sqrt{n}}\left(\frac{e^{-2pt_2^2n/\{\pi(1+q)\}^2}}{|\sin\frac{t_2}{2}|}\right)^{p'}dt_2\right\}^{1/p'}\\
&= O(1)\varphi\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\left\{\int_0^{1/\sqrt{m}}t_1^{-p'}dt_1\int_0^{1/\sqrt{n}}t_2^{-p'}dt_2\right\}^{1/p'}\\
&= O\left\{\varphi\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\cdot(\sqrt{mn})^{1/p}\right\}. \qquad (4)
\end{aligned}$$

Now evaluate $I_2(x,y)$. We have

$$\begin{aligned}
I_2(x,y) &= \int_0^{1/\sqrt{m}}\int_{1/\sqrt{n}}^{\pi}\frac{|\tau(t_1,t_2)|}{\sin\frac{t_1}{2}\sin\frac{t_2}{2}}(1+p)^{-m}(1+q)^{-n}\times\\
&\times \left|\sum_{k=0}^{m}\binom{m}{k}p^{m-k}\sin\left(k+\frac{1}{2}\right)t_1\right|\left|\sum_{l=0}^{n}\binom{n}{l}q^{n-l}\sin\left(l+\frac{1}{2}\right)t_2\right|dt_1dt_2.
\end{aligned}$$

As in [4], page 52 we can find

$$\int_{1/\sqrt{n}}^{\pi}\frac{|\tau(t_1,t_2)|}{\sin\frac{t_2}{2}}(1+q)^{-n}|S(t_2)|dt_2=$$

$$=O\left\{\int_{1/\sqrt{n}}^{\pi}\frac{|\tau(t_1,t_2)|}{\sin\frac{t_2}{2}}(1+q)^{-n}(1+q^2+2q\cos t_2)^{n/2}|R(t_2)|dt_2\right\}.$$

From last estimate and lemma 1 we have

$$\begin{aligned}
I_2(x,y) &= O\left\{\int_0^{1/\sqrt{m}}\int_{1/\sqrt{n}}^{\pi}\frac{|\tau(t_1,t_2)|}{\sin\frac{t_1}{2}\sin\frac{t_2}{2}}\frac{|S(t_1)|}{(1+p)^m}e^{-2qt_2^2n/\{\pi(1+q)\}^2}dt_1dt_2\right\}\\
&= O\left\{\int_0^{1/\sqrt{m}}\int_{1/\sqrt{n}}^{\pi}\frac{\varphi(t_1,t_2)}{(t_1t_2)^{1/p+1}}\frac{1}{t_2}\frac{1}{n}\frac{\partial}{\partial t_2}\left(-e^{-2qt_2^2n/\{\pi(1+q)\}^2}\right)dt_1dt_2\right\}.
\end{aligned}$$

Using Höder's inequality $(p+p'=pp')$ we get

$$\begin{aligned}
I_2(x,y) &= O\Bigg\{\left[\frac{1}{n^p}\int_0^{1/\sqrt{m}}\int_{1/\sqrt{n}}^{\pi}\frac{\varphi^p(t_1,t_2)}{(t_1t_2)^{1+p}\,t_2^p}dt_1dt_2\right]^{1/p}\times \\
&\times \left[\int_0^{1/\sqrt{m}}\int_{1/\sqrt{n}}^{\pi}\left(\frac{\partial}{\partial t_2}\left(-e^{-2qt_2^2n/\{\pi(1+q)\}^2}\right)\right)^{p'}dt_1dt_2\right]^{1/p'}\Bigg\} \\
&= O\left\{\varphi\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\cdot\frac{(\sqrt{mn})^{1/p}}{\sqrt{m}}\right\}. \qquad (5)
\end{aligned}$$

In analogous way we can prove these estimates

$$I_3(x,y)=O\left\{\varphi\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\cdot\frac{(\sqrt{mn})^{1/p}}{\sqrt{n}}\right\}, \qquad (6)$$

and

$$I_4(x,y)=O\left\{\varphi\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\cdot\left(\sqrt{mn}\right)^{1/p-1}\right\}. \qquad (7)$$

From (3), (4), (5), (6) and (7) we shall have

$$\begin{aligned}
&|E_{mn}^{pq}[f](x,y)-f(x,y)| = \\
&= O\left\{\varphi\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\left(\sqrt{mn}\right)^{1/p-1}\left(2\sqrt{mn}+\sqrt{m}+\sqrt{n}\right)\right\} \\
&= O\left\{\varphi\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)(mn)^{1/2p}\right\}.
\end{aligned}$$

The proof of theorem is complete.

Remark 1. If we give value $t_1^{\alpha}\cdot t_2^{\beta}$, $0<\alpha<1$, $0<\beta<1$, to the function $\varphi(t_1,t_2)$, immediately from our theorem we find an interesting estimate

$$\max_{0\le x,y\le 2\pi}|E_{mn}^{pq}[f](x,y)-f(x,y)|=O\left(m^{\frac{1}{2}\left(\frac{1}{p}-\alpha\right)}\cdot n^{\frac{1}{2}\left(\frac{1}{p}-\beta\right)}\right).$$

2.2 n-dimensional case

Let $\{S_{m_1m_2\ldots m_n}\}$ be the sequence of partial sums of the given series $\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\ldots\sum_{m_n=0}^{\infty}u_{m_1m_2\ldots m_n}$. Then for $q_1,q_2,\ldots,q_n>0$ the Euler $(E,q_1,q_2,\ldots,q_n)$ means of sequence $\{S_{m_1m_2\ldots m_n}\}$ we define by

$$E_{m_1m_2\ldots m_n}^{q_1q_2\ldots q_n}=\sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2}\ldots\sum_{k_n=0}^{m_n}\prod_{i=1}^{n}(1+q_i)^{-m_i}\binom{m_i}{k_i}q_i^{m_i-k_i}S_{k_1k_2\ldots k_n}.$$

The series $\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\ldots\sum_{m_n=0}^{\infty}u_{m_1m_2\ldots m_n}$ is said to be $(E,q_1,q_2,\ldots,q_n)$ summable to number S if the sequence $\{E_{m_1m_2\ldots m_n}^{q_1q_2\ldots q_n}\}$ converges to S as m_1,m_2,. . ., $m_n\to\infty$.

Let $f(x_1,x_2,\ldots,x_n)$ be 2π periodic function by each variable and integrable and let $S_{m_1m_2\ldots m_n}(f)$ be the sequence of partial sums of its Fourier series.

The Euler mean $E_{m_1m_2\ldots m_n}^{q_1q_2\ldots q_n}[f]$ of the partial sums of multiple Fourier series of a function$f$ defined in $Q=[-\pi,\pi]^n$ and 2π-periodic in each variable we define by the formula

$$\begin{aligned}&E_{m_1m_2\ldots m_n}^{q_1q_2\ldots q_n}[f](x_1,x_2,\ldots,x_n)=\\=&\sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2}\ldots\sum_{k_n=0}^{m_n}\prod_{i=1}^{n}(1+q_i)^{-m_i}\binom{m_i}{k_i}q_i^{m_i-k_i}S_{k_1k_2\ldots k_n}[f](x_1,x_2,\ldots,x_n),\end{aligned}$$

while for the partial sums of multiple Fourier series the following equality is well-known

$$\begin{aligned}&S_{k_1k_2\ldots k_n}[f](x_1,x_2,\ldots,x_n)=\\=&\frac{1}{\pi^n}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}f(x_1+t_1,x_2+t_2,\ldots,x_n+t_n)\prod_{i=1}^{n}D_{m_i}(t_i)dt_1dt_2\ldots dt_n.\end{aligned}$$

A 2π periodic function $f(x_1,x_2,\ldots,x_n)$ in each variable $x_1,x_2,\ldots,x_n$ is said to belongs to the class $Lip(\varphi(t_1,t_2,\ldots,t_n);p),p>1$, if for $0<t_i\le\pi,i=1,2,\ldots,n$

$$|f(x_1+t_1,x_2+t_2,\ldots,x_n+t_n)-f(x_1,x_2,\ldots,x_n)|\le M\frac{\varphi(t_1,t_2,\ldots,t_n)}{(t_1t_2\ldots t_n)^{1/p}},$$

where $\varphi(t_1,t_2,\ldots,t_n)$ is a positive increasing function of the variables $t_1,t_2,\ldots,t_n$ and M is a positive constant indenpendent of $x_1,x_2,\ldots,x_n$ and $t_1,t_2,\ldots,t_n$.

We put notation

$$\begin{aligned}\tau(t_1,t_2,\ldots,t_n)&=&&\frac{1}{2^n}\{f(x_1+t_1,x_2+t_2,\ldots,x_n+t_n)\\&+&&f(x_1-t_1,x_2+t_2,\ldots,x_n+t_n)+\ldots\\&+&&f(x_1-t_1,x_2-t_2,\ldots,x_n-t_n)-2^nf(x_1,x_2,\ldots,x_n)\}.\end{aligned}$$

Now as a generalization of theorem 1 we can formulate it in n-dimensional case .

Theorem 2. Let $f(x_1, x_2, \ldots, x_n)$ be 2π-periodic function by each variable and belongs to the class $Lip(\varphi(t_1, t_2, \ldots, t_n); p)$, for $p > 1$. If are satisfied 2^n conditions

$$\int_0^{\frac{1}{\sqrt{m_1}}} \int_0^{\frac{1}{\sqrt{m_2}}} \cdots \int_0^{\frac{1}{\sqrt{m_n}}} \frac{\varphi^p(t_1, t_2, \ldots, t_n)}{t_1 t_2 \ldots t_n} dt_1 dt_2 \ldots dt_n =$$

$$= O\left\{\varphi^p\left(\frac{1}{\sqrt{m_1}}, \frac{1}{\sqrt{m_2}}, \ldots, \frac{1}{\sqrt{m_n}}\right)\right\},$$

$$\int_{\frac{1}{\sqrt{m_1}}}^{\pi} \int_0^{\frac{1}{\sqrt{m_2}}} \cdots \int_0^{\frac{1}{\sqrt{m_n}}} \frac{\varphi^p(t_1, t_2, \ldots, t_n)}{(t_1 t_2 \ldots t_n)^{1+p}\, t_1^p} dt_1 dt_2 \ldots dt_n =$$

$$= O\left\{\varphi\left(\frac{1}{\sqrt{m_1}}, \frac{1}{\sqrt{m_2}}, \ldots, \frac{1}{\sqrt{m_n}}\right) m_1\right\}^p,$$

$$\vdots$$

$$\int_0^{\frac{1}{\sqrt{m_1}}} \int_0^{\frac{1}{\sqrt{m_2}}} \cdots \int_{\frac{1}{\sqrt{m_n}}}^{\pi} \frac{\varphi^p(t_1, t_2, \ldots, t_n)}{(t_1 t_2 \ldots t_n)^{1+p}\, t_n^p} dt_1 dt_2 \ldots dt_n =$$

$$= O\left\{\varphi\left(\frac{1}{\sqrt{m_1}}, \frac{1}{\sqrt{m_2}}, \ldots, \frac{1}{\sqrt{m_n}}\right) m_n\right\}^p,$$

$$\int_{\frac{1}{\sqrt{m_1}}}^{\pi} \int_{\frac{1}{\sqrt{m_2}}}^{\pi} \cdots \int_0^{\frac{1}{\sqrt{m_n}}} \frac{\varphi^p(t_1, t_2, \ldots, t_n)}{(t_1 t_2 \ldots t_n)^{1+p}\, (t_1 t_2)^p} dt_1 dt_2 \ldots dt_n =$$

$$= O\left\{\varphi\left(\frac{1}{\sqrt{m_1}}, \frac{1}{\sqrt{m_2}}, \ldots, \frac{1}{\sqrt{m_n}}\right) m_1 m_2\right\}^p,$$

$$\vdots$$

$$\int_{\frac{1}{\sqrt{m_1}}}^{\pi} \int_{\frac{1}{\sqrt{m_2}}}^{\pi} \cdots \int_{\frac{1}{\sqrt{m_n}}}^{\pi} \frac{\varphi^p(t_1, t_2, \ldots, t_n)}{(t_1 t_2 \ldots t_n)^{1+2p}} dt_1 dt_2 \ldots dt_n =$$

$$= O\left\{\varphi\left(\frac{1}{\sqrt{m_1}}, \frac{1}{\sqrt{m_2}}, \ldots, \frac{1}{\sqrt{m_n}}\right) \prod_{i=1}^{n} m_i\right\}^p,$$

then

$$\max_{0 \le x_1, x_2, \ldots, x_n \le 2\pi} |E_{m_1 m_2 \ldots m_n}^{q_1 q_2 \ldots q_n}[f](x_1, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)| =$$

$$= O\left\{\varphi\left(\frac{1}{\sqrt{m_1}}, \frac{1}{\sqrt{m_2}}, \ldots, \frac{1}{\sqrt{m_n}}\right) \left(\prod_{i=1}^{n} m_i\right)^{\frac{1}{2p}}\right\}.$$

The proof of theorem 2 is more complicate than theorem 1. However, we observe that theorem 2 of this paper can be proved in the same way.

Remark 2. If we give value $t_1^{\alpha_1} t_2^{\alpha_2} \dots t_n^{\alpha_n}$, $0 < \alpha_i < 1$, $i = 1, 2, \dots, n$, to the function $\varphi(t_1, t_2, \dots, t_n)$, immediately from theorem 2 we have this estimate

$$\max_{0 \leq x_1, x_2, \dots, x_n \leq 2\pi} |E^{q_1 q_2 \dots q_n}_{m_1 m_2 \dots m_n}[f](x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)|$$

$$= O\left(m_1^{\frac{1}{2}\left(\frac{1}{p}-\alpha_1\right)} m_2^{\frac{1}{2}\left(\frac{1}{p}-\alpha_2\right)} \dots m_n^{\frac{1}{2}\left(\frac{1}{p}-\alpha_n\right)}\right).$$

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