Applied Mathematical Sciences, Vol. 3, 2009, no. 8, 365 - 376

New Third Order Runge Kutta Based on Contraharmonic Mean for Stiff Problems

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Abstract

In this paper we introduce an explicit one–step method that can be used for solving stiff problems. This method can be viewed as a modification of the explicit third order runge-kutta method using the contraharmonic mean (C_oM) that allows rducing the stiffiness in some sense. The stability of the method is analyzed and numerical results shown to verify the conclusions.numerical examples indicate that this method is superior compared to some existing methods including the third and fourth order contraharmonic mean methods ,ABM method, classical third order Runge-Kutta , and Wazwaz method.

Mathematics Subject Classifications: 51N20, 62J05, 70F99

Keywords: ODE solver; Runge-Kutta method; contraharmonic mean; stiff problems

1 Introduction

It has been suggested that The problem of stiffness is very difficult to be solved by explicit methods but recently many explicit methods were introduced and developed to solve the stiff problems like [1–3]. In this paper we introduced a Runge-Kutta like explicit method can be used to solve stiff problems and give a good accuracy butter than wide of an explicit methods.

Theorm 1.1 *[4]: If an p–stage explicit Runge-Kutta contraharmonic mean(C_0M) method has order N where $N \leq 3$ then $p \leq N$.*

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As a 2 stage formula of second order the contraharmonic mean(C_0M) method is

$$y_{n+1} = y_n + h\left[\frac{k_1^2 + k_2^2}{k_1 + k_2}\right]$$

where

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f(x_n + h, y_n + hk_1). \end{aligned}$$

A 3rd -order method for 3-stages of the (C_0M) method are given in the form

$$y_{n+1} = y_n + \frac{h}{2}\left[\frac{k_1^2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_3^2}{k_2 + k_3}\right]$$

where,

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f(x_n + h\tfrac{2}{3}, y_n + h\tfrac{2}{3}k_1), \\ k_3 &= f(x_n + h\tfrac{2}{3}, y_n + h\tfrac{2}{3}k_2). \end{aligned}$$

The fourth order contraharmonic mean (C_0M) method can be expressed in the form

$$y_{n+1} = y_n + \tfrac{h}{3}\left[\tfrac{k_1^2+k_2^2}{k_1+k_2} + \tfrac{k_2^2+k_3^2}{k_2+k_3} + \tfrac{k_3^2+k_4^2}{k_3+k_4}\right]$$

where

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f(x_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}k_1), \\ k_3 &= f(x_n + \tfrac{h}{2}, y_n + \tfrac{1}{8}hk_1 + \tfrac{3}{8}hk_2). \\ k_4 &= f(x_n + h, y_n + \tfrac{1}{4}hk_1 - \tfrac{3}{4}hk_2 + \tfrac{3}{2}k_3). \end{aligned}$$

2 Modified C_0M Weights Runge-Kutta Method (MCHW-RK3)

It is possible to establish a three-stage Runge-Kutta formula based on the contraharmonic mean using the mean in the main formula which can be presented as follows:

$$y_{n+1} = y_n + h\left[w_1\frac{k_1^2 + k_2^2}{k_1 + k_2} + w_2\frac{k_2^2 + k_3^2}{k_2 + k_3}\right] \tag{1}$$

where

$$k_1 = f(x_n, y_n) = f,$$
$$k_2 = f(x_n + ha_1, y_n + ha_1k_1),$$
$$k_3 = f(x_n + h(a_2 + a_3), y_n + h(a_2k_1 + a_3k_2).$$

w_1 and w_2 are the weights chosen in such a way so that a_1 and a_2 are parameters to be determined and $\frac{k_i^2+k_{i+1}^2}{k_i+k_{i+1}}$ is defined as the contraharmonic mean. Notice that for simplicity of the algebra f have been considered as a function of y only, without loss of generality. This will considerably reduce the Taylor series expansions of k_i $i = 1, 2, 3$, to the following

$$k_1 = f \tag{2}$$

$$k_2 = f + ha_1ff_y + \frac{1}{2}f^2a_1^2h^2f_{yy} + \frac{1}{6}f^3h^3a_1^3f_{yyy} + ... \tag{3}$$

$$k_3 = f + h(a_2 + a_3)ff_y + h^2(a_1a_3ff_y^2 + \frac{1}{2}(a_2 + a_3)^2f^2f_{yy})+ \tag{4}$$

$$h^3(\frac{1}{2}a_1^2a_3f^2f_yf_{yy} + a_1a_3(a_2 + a_3)f^2f_yf_{yy} + \frac{1}{6}(a_2 + a_3)^3f^3f_{yyy}) + ...$$

Traditionally, the equations (2)-(4) would be substituted to obtain an expression of y_{n+1}in terms of the function together with the parameters a_i, $i = 2, 3$ and its derivatives. Since the algebra involved is the division of two series,

$$\frac{k_i^2 + k_{i+1}^2}{k_i + k_{i+1}}, i = 1(1)3 \tag{5}$$

direct substitution cannot be done. These problems are alleviated by multiplying the terms across with the common denominator$(k_1 + k_2)(k_2 + k_3)$and can be written as

$$y_{n+1} = y_n + \frac{upper}{lower} \tag{6}$$

with

$$upper = h(w_1(k_1^2 + k_2^2)(k_2 + k_3) + w_2(k_2^2 + k_3^2)(k_1 + k_2))$$

and

$$lower = (k_1 + k_2)(k_2 + k_3)$$

Taylor series expansion of $y(x_{n+1})$ may be written as

$$\text{Taylor} = y_n + hf + \frac{1}{2}h^2 ff_y + \frac{1}{6}h^3(ff_y^2 + f^2 f_{yy}) + \qquad (7)$$

$$+\frac{1}{24}h^4(f^3 f_{yyy} + 4f^2 f_y f_{yy} + ff_y^3) + ..$$

Since the error of the method can be measured using the expression

$$\text{Error} = y(x_{n+1}) - y_{n+1}$$

thus, we get

$$\text{Error} = \text{Taylor} - \frac{\text{upper}}{\text{lower}}$$

which could be written as

$$\text{Error} \times \text{lower} = \text{Taylor} \times \text{lower} - \text{upper} \qquad (8)$$

Comparing the coefficients of the same terms in(8) up to the term h^3,the following equations of conditions were obtained:

$$f^2 h : -4w_1 - 4w_2 + 4 = 0 \qquad (9)$$

$$h^2 f^3 f_y : -6a_1w_1 - 6a_1w_2 - 2a_2w_1 - 4a_2w_2 - \qquad (10)$$

$$2a_3w_1 - 4a_3w_2 + 2 + 4a_1 + 2a_2 + 2a_3 = 0$$

$$h^3 f^3 f_y^2 : -4a_1^2w_1 - 4a_1^2w_2 - 2a_1a_2w_1 - 2a_1a_2w_2 - 2a_2^2w_2 - 4a_1a_3w_1 \qquad (11)$$

$$-6w_2a_1a_3 - 6w_2a_1a_3 - 4w_2a_2a_3$$

$$h^3 f^4 f_{yy} : -3a_1^2w_1 - 3a_1^2w_2 - a_2^2w_1 - 2a_2^2w_2 - 2a_2a_3w_1 - 4a_2a_3w_2 \qquad (12)$$

$$-a_3^2w_1 - 2w_2a_3^2 + \frac{1}{3}(2 + 6a_1^2 + 3a_2^2 + 6a_2a_3 + 3a_3^2) = 0$$

Solving equations (9)-(12) using MATHEMATICA [5] we obtained a set of parameters and weights a shown below.

$$w_2 = \frac{3}{4}, w_1 = \frac{1}{4}, a_3 = \frac{4}{21}(3 + \sqrt{2})$$

$$a_2 = 0, a_1 = \frac{1}{7}(4 - \sqrt{2})$$

The third order contraharmonic mean.RK formula MCHW-RK3 can be represented by

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f(x_n + h\tfrac{2}{3}, y_n + h\tfrac{2}{3}k_1), \\ k_3 &= f(x_n + h\tfrac{4}{21}(3 + \sqrt{2}), y_n + h\tfrac{4}{21}(3 + \sqrt{2})k_2). \end{aligned} \tag{13}$$

$$y_{n+1} = y_n + h\left[\frac{1}{4}\frac{k_1^2 + k_2^2}{k_1 + k_2} + \frac{3}{4}\frac{k_2^2 + k_3^2}{k_2 + k_3}\right] \tag{14}$$

3 Stability Analysis

To check on the stability, the equations in(13) and(14) are substituted into the simple test equation proposed by Dahlquist[6] is used $y' = \lambda y$ and it yields

$$k_1 = f(x_n, y_n) = \lambda y_n \tag{15}$$

$$k_2 = f(x_n + h\frac{1}{7}(4 - \sqrt{2}), y_n + h\frac{1}{7}(4 - \sqrt{2})h\lambda y_n) \tag{16}$$

$$= \lambda(y_n + \frac{1}{7}(4 - \sqrt{2})h\lambda y_n) = \lambda y_n(1 + \frac{1}{7}(4 - \sqrt{2})h\lambda)$$

$$k_3 = f(x_n + h\frac{4}{21}(3 + \sqrt{2}), y_n + h\frac{4}{21}(3 + \sqrt{2})\lambda y_n(1 + \frac{1}{7}(4 - \sqrt{2})h\lambda)) \tag{17}$$

$$= \lambda(y_n + h\frac{4}{21}(3 + \sqrt{2})\lambda y_n(1 + \frac{1}{7}(4 - \sqrt{2})h\lambda))$$

$$= \lambda y_n(1 + h\frac{4}{21}(3 + \sqrt{2})\lambda(1 + \frac{1}{7}(4 - \sqrt{2})h\lambda))$$

Substituting (15), (16) and (17) in (14), and letting $z = h\lambda$, we obtain the simplified equation

$$y_{n+1} = y_n + h\left[\frac{1}{4}\frac{\lambda^2 y_n^2\left(2+\frac{8}{7}z-\frac{2}{7}\sqrt{2}z+\frac{18}{49}z^2-\frac{8}{49}\sqrt{2}z^2\right)}{\lambda y_n\left(1+(1+\frac{1}{7}(4-\sqrt{2})z\right)} + \right.$$
$$\left. \frac{3}{4}\ \frac{\lambda^2 y_n^2\left(1+\frac{8}{7}z+\frac{8}{21}\sqrt{2}z+\frac{416}{441}z^2+\frac{40}{147}\sqrt{2}z^2+(\frac{1024+416\sqrt{2}}{3087})z^3)+\frac{544}{7203}z^4+\frac{320\sqrt{2}}{21609}z^4\right)}{\lambda y_n\left(\left(1+\frac{1}{7}(4-\sqrt{2})z\right)+\left(1+\frac{4}{21}(3+\sqrt{2})z\left(1+\frac{1}{7}(4-\sqrt{2})z\right)\right)\right)}\right]$$

$$y_{n+1} = y_n + h\left[\frac{1}{4}\frac{\lambda y_n\left(2+\frac{8}{7}z-\frac{2}{7}\sqrt{2}z+\frac{18}{49}z^2-\frac{8}{49}\sqrt{2}z^2\right)}{\left(1+(1+\frac{1}{7}(4-\sqrt{2})z\right)}+\right.$$
$$\left.\frac{3}{4}\,\frac{\lambda y_n\left(1+\frac{8}{7}z+\frac{8}{21}\sqrt{2}z+\frac{416}{441}z^2+\frac{40}{147}\sqrt{2}z^2+(\frac{1024+416\sqrt{2}}{3087})z^3+\frac{544}{7203}z^4+\frac{320\sqrt{2}}{21609}z^4\right)}{\left(\left(1+\frac{1}{7}(4-\sqrt{2})z\right)+\left(1+\frac{4}{21}(3+\sqrt{2})z\left(1+\frac{1}{7}(4-\sqrt{2})z\right)\right)\right)}\right]$$

$$y_{n+1} = y_n + zy_n\left[\frac{1}{4}\frac{\left(2+\frac{8}{7}z-\frac{2}{7}\sqrt{2}z+\frac{18}{49}z^2-\frac{8}{49}\sqrt{2}z^2\right)}{\left(1+(1+\frac{1}{7}(4-\sqrt{2})z\right)}+\right.$$
$$\left.\frac{3}{4}\,\frac{\left(1+\frac{8}{7}z+\frac{8}{21}\sqrt{2}z+\frac{416}{441}z^2+\frac{40}{147}\sqrt{2}z^2+(\frac{1024+416\sqrt{2}}{3087})z^3+\frac{544}{7203}z^4+\frac{320\sqrt{2}}{21609}z^4\right)}{\left(\left(1+\frac{1}{7}(4-\sqrt{2})z\right)+\left(1+\frac{4}{21}(3+\sqrt{2})z\left(1+\frac{1}{7}(4-\sqrt{2})z\right)\right)\right)}\right]$$

which yield the stability polynomial:

$$y_{n+1} = y_n(1 + z\left[\frac{1}{4}\frac{\left(2+\frac{8}{7}z-\frac{2}{7}\sqrt{2}z+\frac{18}{49}z^2-\frac{8}{49}\sqrt{2}z^2\right)}{\left(1+(1+\frac{1}{7}(4-\sqrt{2})z\right)}+\right.$$
$$\left.\frac{3}{4}\,\frac{(1+\frac{8}{7}z+\frac{8}{21}\sqrt{2}z+\frac{416}{441}z^2+\frac{40}{147}\sqrt{2}z^2+(\frac{1024+416\sqrt{2}}{3087})z^3)+\frac{544}{7203}z^4+\frac{320\sqrt{2}}{21609}z^4)}{\left(\left(1+\frac{1}{7}(4-\sqrt{2})z\right)+\left(1+\frac{4}{21}(3+\sqrt{2})z\left(1+\frac{1}{7}(4-\sqrt{2})z\right)\right)\right)}\right])$$

or in more simplified form,

$$y_{n+1} = y_n R(z)$$

where

$$R(z) = 1 + z\left[\frac{1}{4}\frac{\left(2+\frac{8}{7}z-\frac{2}{7}\sqrt{2}z+\frac{18}{49}z^2-\frac{8}{49}\sqrt{2}z^2\right)}{\left(1+(1+\frac{1}{7}(4-\sqrt{2})z)\right)}+\right.$$
$$\left.\frac{3}{4}\,\frac{\left(1+\frac{8}{7}z+\frac{8}{21}\sqrt{2}z+\frac{416}{441}z^2+\frac{40}{147}\sqrt{2}z^2+(\frac{1024+416\sqrt{2}}{3087})z^3+\frac{544}{7203}z^4+\frac{320\sqrt{2}}{21609}z^4\right)}{\left(\left(1+\frac{1}{7}(4-\sqrt{2})z\right)+\left(1+\frac{4}{21}(3+\sqrt{2})z\left(1+\frac{1}{7}(4-\sqrt{2})z\right)\right)\right)}\right]$$

The stability region for the above formula is illustrated in the graphs below:

4 Stiff problem

4.1 Definition[8]: *if a numerical method is forced to use, in an interval of integration, a stepsize is forced to be excessively small relative to dominant time-scale of the solution to get a smooth approximation of the exact solution in that interval,then the problem is said to be stiff in that interval.*

According to Definition 4.1, in order to get a smooth approximation of the solution need to use very small stepsize for stiff problems. In practice use large stepsize to reduce computational costs.

5 Numerical Experiments

The MCHW–RK3 is tested on two examples of ordinary differential equations which are stiff problems to check on the accuracy of this method,we will comparing the new method by some existing methods with defrrent h and n,including the third and fourth order contraharmonic mean (C_0M) methods[4],Adam Bashforth Moulton method method(ABM), classical third order Runge-Kutta ,Nystrom method and Waswas method[7], where the third order classical Runge-Kutta method uses the formula

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f(x_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}k_1), \\ k_3 &= f(x_n + h, y_n - hk_1 + 2hk_2). \end{aligned}$$

where

$$y_{n+1} = y_n + \frac{h}{6}\left[k_1 + 4k_2 + k_3\right]$$

Example 1: Consider the stiff ordinary differential equation

$$y'(t) = -100y(t) + e^{-2t}; y(0) = 0$$

with the exact solution

$$y(t) = \frac{1}{98} + e^{-100t}(-1 + e^{98t})$$

was considered over the range $0 \le t \le 1$ using a stepsize $h = 0.01$, and $h = 0.02$

6 Discussion and Conclusion

The research done in this paper shows the possibility of constructing method out of the various forms of explicit three-stage third order contraharmonic mean Runge-Kutta formula to solve stiff problem. With the purpose of verifying the accuracy, the example used is solved using the MCHW-RK3 method and some existing methods.Figures 1, 2 and 3 demonstrate the stability for the new method, the results show excellent accuracy of the MCHW–RK3 method using tow step sizes ,h=0.01 and h=0.02 which are better than all the methods results unless classical RK4 method ,nevertheless classical RK4 method shows a better results when h=0.01 compared to h=0.02 which shows that this method requires more iteration to obtain a better accuracy and same thing for the third order CH-RK3 ,but the new method gives abetter accuracy than calssical RK4 and CH-RK3 when h=0.02 Table 1 shows the numerical solutions and the absolute errors of the example for the methods when h=0.01 whilst Table 2 when h=0.02. The results of using the MCHW–RK3 method and the other methods for solving this stiff problem on the interval [0,1] are presented in Figures 4,5,6,7,8,9 and 10. Figures 6 and 7 show that the classical RK3 and RK4 methods definitely seems to have difficulty of the aproximation when h=0.02 but without difficulties with h=0.01.However, the MCHW–RK3 method performs perfectly well, even for h=0.01 and h=0.02 as shown in Figure 4 (a) and (b).in Figures 7–10 the computed solutions of this problem with h=0.02 using the third and fourth order contraharmonic mean (C_0M) methods(CH-RK3 and CH-RK4),Adam Bashforth Moulton method(ABM) and Waswas methed are performed poorly. from this discussion it is clearly confirmed that the MCHW–RK3 method is appropriate for stiff problems.

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Received: July, 2008

Table 1: the absolute error of the Explicit MCHW-RK3 method, h=0.01 on the equations y' = -100y(t)+exp(-2t) compared to the RK3,RK4,C_0M3,C_0M4, ABM and WAZWAZ methods

	Y–EXACT	MCHW3	RK3	RK4	C_0M3	C_0M4	ABM	WAZWAZ
2	0.0083539322	5.56E-08	7.59E-08	4.10E-08	4.13E-07	5.75E-08	3.03E-05	3.18E-04
4	0.0068400004	1.26E-07	1.76E-07	4.67E-08	7.28E-08	3.23E-07	4.48E-06	2.66E-04
6	0.0056001287	9.33E-08	1.54E-07	2.82E-08	4.96E-08	2.54E-07	4.16E-08	2.18E-04
8	0.0045849994	7.45E-08	1.28E-07	2.13E-08	3.88E-08	2.06E-07	8.39E-08	1.79E-04
10	0.0037538718	6.93E-08	9.63E-08	2.56E-08	4.00E-08	1.77E-07	7.20E-09	1.46E-04
12	0.0030734103	5.66E-08	7.89E-08	2.10E-08	3.27E-08	1.45E-07	4.00E-10	1.20E-04
14	0.0025162956	4.63E-08	6.47E-08	1.71E-08	2.67E-08	1.19E-07	2.00E-10	9.81E-05
16	0.0020601786	2.79E-08	6.30E-08	4.00E-09	1.19E-08	8.72E-08	1.00E-08	8.03E-05
18	0.0016867233	3.12E-08	4.33E-08	1.15E-08	1.80E-08	7.96E-08	0.00E+00	6.57E-05
18	0.0013809723	2.54E-08	3.55E-08	9.40E-09	1.46E-08	6.52E-08	0.00E+00	5.38E-05

Table 2: the absolute error of the Explicit MCHW-RK3 method, h =0.02 on the equations y' = -100y(t)+(-2t) compared to the RK3,RK4,C_0M3,C_0M4, ABM and WAZWAZ methods

	Y–EXACT	MCHW3	RK3	RK4	C_0M3	C_0M4	ABM	WAZWAZ
2	0.0083539322	5.67E-07	4.07E-05	3.98E-05	1.37E+00	1.37E+00	2.88E-03	1.68E+00
4	0.0068400004	1.51E-07	1.57E-06	1.26E-06	1.77E+02	1.77E+02	8.28E-03	2.62E+02
6	0.0056001287	1.13E-07	1.16E-06	1.16E-06	2.29E+04	2.29E+04	1.93E-02	4.07E+04
8	0.0045849994	1.02E-03	9.49E-07	9.52E-07	2.97E+00	2.97E+00	2.23E-02	6.33E+00
10	0.0037538718	8.26E-08	7.68E-07	7.87E-07	3.83E+00	3.83E+00	6.23E-01	9.83E+00
12	0.0030734103	6.76E-08	6.29E-07	6.45E-07	4.96E+00	4.96E+00	4.76E+00	1.53E+00
14	0.0025162956	5.53E-08	5.15E-07	5.28E-07	6.40E+00	6.40E+00	2.62E+01	2.38E+00
16	0.0020601786	3.53E-08	4.32E-07	4.22E-07	8.28E+00	8.28E+00	1.15E+02	3.69E+00
18	0.0016867233	3.71E-08	3.45E-07	3.54E-07	1.07E+00	1.07E+00	3.83E+02	5.74E+00
18	0.0013809723	3.03E-08	2.83E-07	2.90E-07	1.39E+00	1.39E+00	6.53E+02	8.92E+00

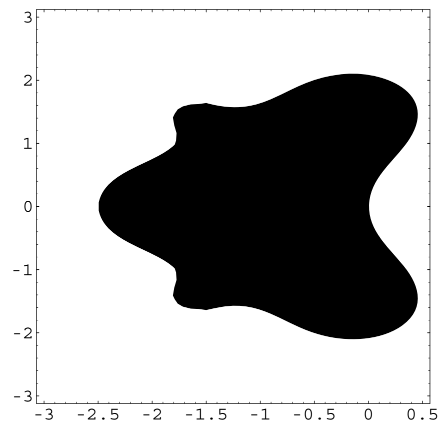

Figure 1: The stability region of the MCHW-RK3 method in 2D.

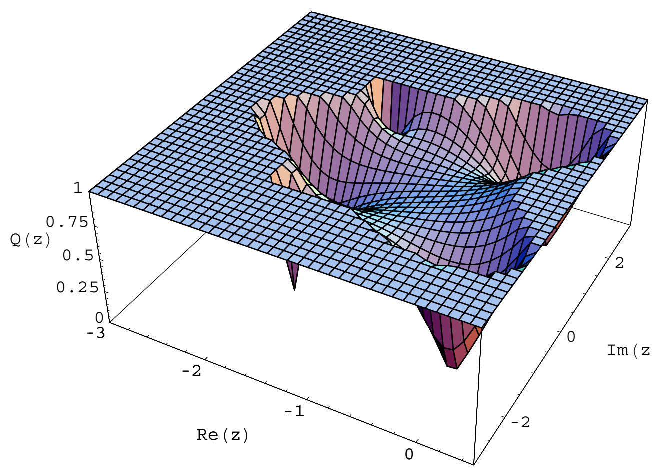

Figure 2: The stability region of the MCHW-RK3 method in 3D.

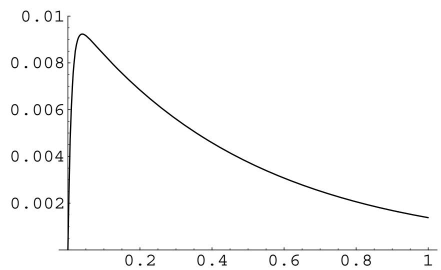

Figure 3: Analytical solution of Example 1

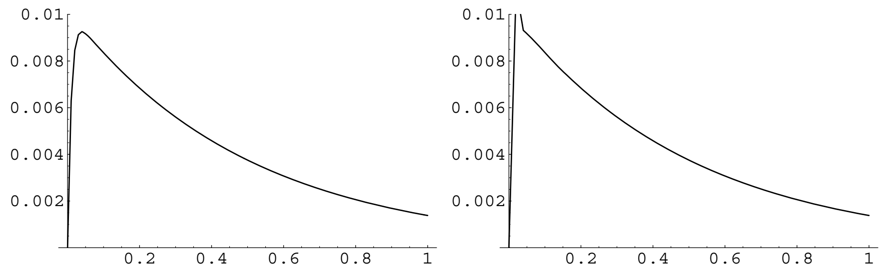

Figure 4: The MCHW-RK3 method used to solve the stiff problem for(a) h = 0.01 and (b) h=0.02.

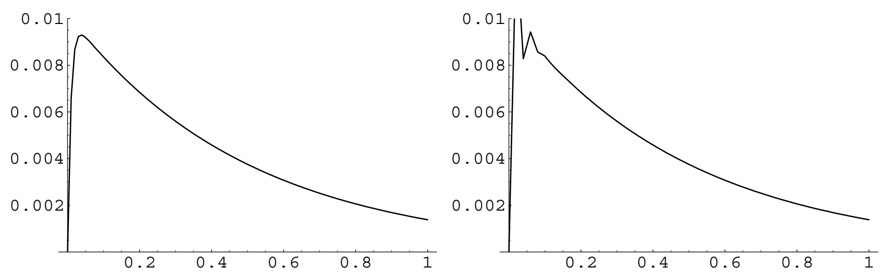

Figure 5: The RK3 method used to solve the stiff problem for(a) h = 0.01 and (b) h=0.02.

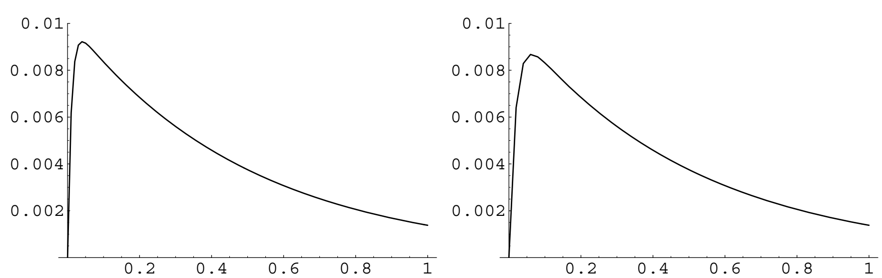

Figure 6: The RK4 method used to solve the stiff problem for (a) h = 0.01 and (b) h=0.02.

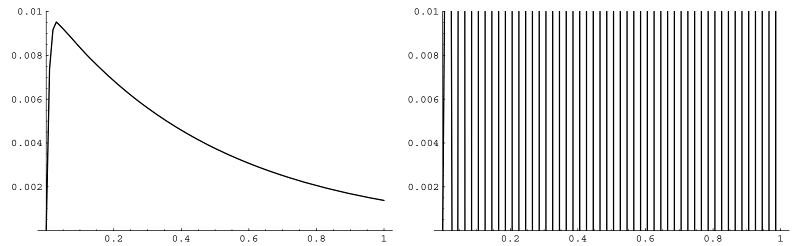

Figure 7: The $(C_oM)3$ method used to solve the stiff problem for(a)h = 0.01 and(b) h=0.02.

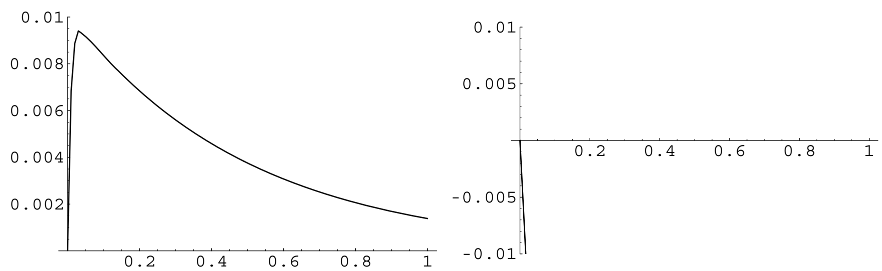

Figure 8: he $(C_oM)4$ method used to solve the stiff problem for(a)h = 0.01 and(b)h=0.02.

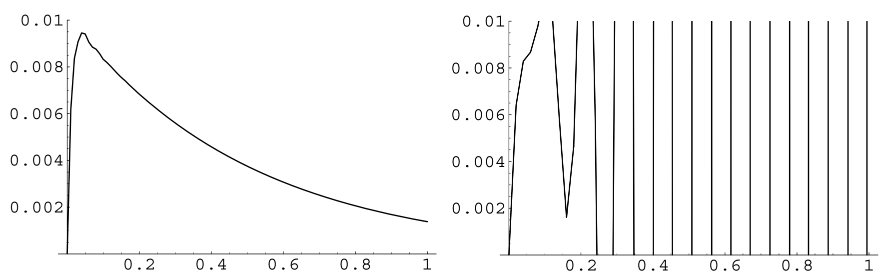

Figure 9: The ABM method used to solve the stiff problem for(a) h=0.01and (b)h=0.02.

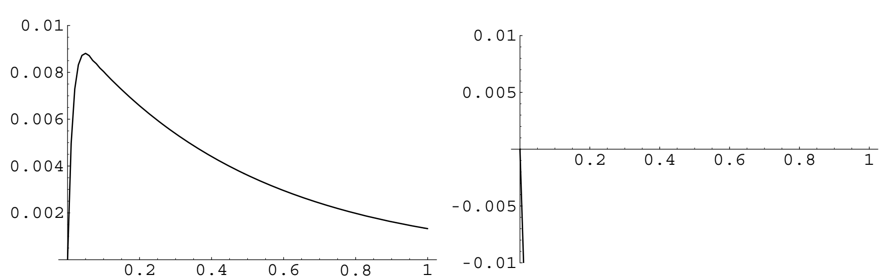

Figure 10: The WAZWAZ method used to solve the stiff problem for(a) h =0.01and(b)h=0.02.