A Generalized Sensitivity Analysis DEA Model

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Abstract

Sensitivity analysis in DEA is used for improving the efficiency scores of inefficient DMUs for which the efficient units remain unchanged. This paper introduces a generalized sensitivity analysis DEA model by perturbation a given input (or output) for all efficient DMUs. A numerical example illustrates the usefulness of the new model.

Keywords: Data envelopment analysis, Sensitivity analysis, Stability radius, Efficiency

1. Introduction

The data envelopment analysis (DEA) technique, first proposed by Charnes et al. (1978) and extended by Banker et al. (1984), is now widely applied for measuring of the efficiency of many entities (such as schools, public agencies, banks, etc). During the recent years, the issue of sensitivity and stability of DEA models has been extensively studied. By updating the inverse of an optimal basis matrix Charnes et al. (1985) discussed the sensitivity of the original DEA model. Also, Charnes and Neralic (1990) investigated the sensitivity of DEA-additive model. Zhu et al. (1998) proposed models to find the stability radius for an efficient DMU. Those authors (Zhu et al. 1998) considered the percentage and absolute changes of a given group of inputs and outputs of Decision Making Units (DMUs). Also, Metters et al. (2001) examined the stability in a DMU category. They have, they partitioned DMUs, and then determined the stability of DMUs

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using a trial and error unit scheme. Jahanshahloo et al. (2004) proposed models to find the stability radius of each unit in the presence case of, interval data in such a way that, the classification remains unchanged. This paper intends to introducing a new sensitivity DEA model that perturbations a given input (or output) of efficient DMUs. This generalizes the existing sensitivity DEA models. This model finds a stability radius that preserves all efficient units unchanged, and generalizes the existing literature. The rest of this paper is organized as follows: gives the necessary preliminaries, Section 2. A model for determining the stability radius for a given group of efficient DMUs is introduced in Section 3. Section 4 shows a numerical illustration for the presented method. The conclusion remarks and further research are given in Section 5.

2. Motivation

Assume that we have \( n \) DMUs, where \( DMU_j \ (j=1,...,n) \) consumes \( x_j = (x_{j1},...,x_{jm})' \) as inputs produces \( y_j = (y_{j1},...,y_{jk})' \). For evaluation of DMU\(_k\) under constant return to scale the following linear programming model, CCR model, should be solved

\[
\begin{align*}
 h_k^* &= \max \ u y_k \\
 s.t. \\
 u y_j - v x_j &\leq 0 \quad j = 1,...,n \\
 v x_k &= 1 \\
 u &\geq 0, \quad v \geq 0
\end{align*}
\]

Where, \( u = (u_1,...,u_j) \) and \( v = (v_1,...,v_m) \) are defined as the vectors of weights of outputs and inputs, respectively. We denote \( E \) and \( \hat{E} \) as the sets of efficient and inefficient DMUs, respectively. Also let \( E^* \) shows the set of efficient units distinguished by model (1) if we ignore the efficient DMUs given in \( E \), as shown in the following figure. We define set \( E \) and \( E^* \) as the first and second levels efficient units, respectively.
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Figure 1 indicates a two-dimensional production possibility set (PPS) consisting four DMUs, A, B, C and D. Also it shows the first and second CCR efficiency frontiers. As it is shown A is CCR efficient unit, \( E = \{ A \} \), and \( E^* = \{ B, C \} \). Figure 2 shows the motivation of this study. As the figure indicates we are looking for a suitable perturbation of the input of unit A as follows: Increase the input of DMU A to \( x_A (1 + \beta) \) and decrease the input of the remaining units to \( x_j (1 - \beta), j = B, C, D \), (without changing output), in such a way that the efficient unit A remains unchanged and the efficiency scores of the other units are improved (if it is possible). As Figure 2 shows we are looking for a maximum value of \( \beta \). In this case one of the second layer efficient units, B or C, becomes to the set of new efficient unit. More precisely, with this sensitivity perturbation one can improve the efficiency values of some inefficient unit without worsening the score of efficient units.

3. Finding a new stability radius

In this section we propose a new percentage of changes that increases a given input (or decreases a given output) of all DMUs in \( E \) and decreases the input (or increases the output) of all DMUs in \( E^* \). Our objective is to find the maximum perturbation for which all of the first level efficient DMUs remain unchanged. We propose a model that combines \( p \), CCR models, by introducing the following constraints:

\[
\begin{align*}
\sum_{r=1}^{s} u_r y_{rk} - \left( \sum_{i=1, i \neq k}^{m} v_r x_{rk} + v_r (x_{rk} (1 + \beta)) \right) & \leq 0 \hspace{1cm} \forall k \in E \\
\sum_{r=1}^{s} u_r y_{rj} - \left( \sum_{i=1, i \neq j}^{m} v_r x_{rj} + v_r (x_{rj} (1 - \beta)) \right) & \leq 0 \hspace{1cm} \forall j \in E^*
\end{align*}
\]
Where, \( p = |E| \geq 1 \) and \( o \) denotes the index of under investigation input. Also we insert the constraint \( \sum_{k \in E} \sum_{r=1}^{s} u_{i} y_{rk} = p \) for all efficient units \( k \in E \).

For simplicity, we denote this perturbation by defining the following sets:

\[ X_i = \{ (U_i, V_i, \beta_o) \mid V_i X_i + v_{lo} \beta_o = 1, U_i Y_i - (\sum_{i=1}^{m} v_{oi} x_{ik} + v_{lo} (x_{ik} (1+\beta_o))) \leq 0, \forall k \in E \} \]

\[ U_i Y_i - (\sum_{i=1}^{m} v_{oi} x_{ij} + v_{lo} (x_{ij} (1-\beta_o))) \leq 0, \forall j \in E^c \] and \( \beta_o \geq 0, U_i \geq 0, V_i \geq 0 \}

Where \( U_i = (u_{i1}, \ldots, u_{im}) \)

\( V_i = (v_{i1}, \ldots, v_{im}) \)

Now we propose the following model

\[ \beta_o^* = \max \beta_o \]

s.t. \((U_i, V_i, \beta_o) \in X_i, \ l \in E \)

\[ \sum_{k \in E} \sum_{r=1}^{s} u_{i} y_{rk} = p \]

Clearly model (2) is nonlinear. To linearize, we define

\[ \overline{X}_i = \{ (\overline{U}_i, \overline{V}_i, \beta_o) \mid (U_i, V_i, \beta_o) \in X_i, \ \forall l \in E \}

Where \( \overline{U}_{il} = \frac{U_{il}}{v_{lo}} \) & \( \overline{U}_{ir} = \frac{U_{ir}}{v_{lo}} \) & \( \overline{V}_{lo} = \frac{1}{v_{lo}} (i=1,\ldots,m, l \neq o \ \& \ r=1,\ldots,s \ \& \ \forall l \in E) \)

So model (2) can be written as the following linear programming problem.

\[ \beta_o^* = \max \beta_o \]

s.t. \((\overline{U}_i, \overline{V}_i, \beta_o) \in \overline{X}_i, \ \forall l \in E \)

\[ \sum_{i \in E} \sum_{r=1}^{s} \overline{U}_{ir} Y_{rk} = \sum_{i=1}^{s} \overline{V}_{lo} \]

The optimal value of model (3), \( \beta_o^* \), shows the reduction level of the \( o^\text{th} \) input for each efficient unit. It also improves the efficiency scores of some inefficient units by preserving the efficient units. Similar model can be introduced by adding \( \beta \geq 0 \), in \( x_{ko} \) (for each \( k \in E \)) and \(-\beta\) in \( x_{jo} \) (for each inefficient unit \( j \in E^c \)). Note that model (3) is unbounded if and only if the input of efficient DMUs can be infinitely increased (or the selected output can be infinitely decreased). This assumption produces the case of output stability in a similar manner. Now we show the feasibility of the proposed model (3).

**Theorem 1.** Model (3) is feasible.

**Proof.** Let \((U_k^*, V_k^*)\) denotes an optimal solution of the corresponding CCR model, for each DMU \( k \in E \), then \((U_i^*, V_i^*, \beta_o) \in X_i \) by taking \( \beta_o = 0 \). Similarly \((U_k^*, V_k^*, 0) \in \overline{X}_i \). This completes the proof. \( \checkmark \)

**Definition 1:** \( \beta \) is called the maximum stability radius for the \( o^\text{th} \) input \(( o = 1,\ldots,m )\) if for the input changes
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\[ \forall \alpha \in [0, \beta], (x_k (1+\alpha)), (\forall k \in E) \text{ and } (x_{j\alpha} (1-\alpha)), (\forall j \in E^*) \]

the efficient units remain unchanged, and for the following changes

\[ \forall \gamma (\gamma > \beta), (x_k (1+\gamma)), (\forall k \in E) \text{ and } (x_{j\gamma} (1-\gamma)), (\forall j \in E^*) \]

at least one first level efficient unit becomes inefficient one.

Theorem 2. \( \beta^*_\alpha \) is the maximum stability radius for the \( \alpha^{th} \) input, \( \alpha = 1, \ldots, m \), where \( \beta^*_\alpha \) is the optimal objective value of model (3).

Proof. By contradiction, suppose there is \( \beta \) such that \( \beta > \beta^*_\alpha \) and \( h_k^* = 1, \forall k \in E \).

Since \( h_k^* = 1 \), the efficiency score of the \( k^{th} \) DMU \( (k \in E) \), there would be exist

\[ v_{ij}u_{ij} (i = 1, \ldots, m, r = 1, \ldots, s) \]

such that \( \beta, v_{ij} \) and \( u_{ij} \) \( (i = 1, \ldots, m, r = 1, \ldots, s) \) are a feasible solution of model (3). Therefore we conclude that \( \beta \leq \beta^*_\alpha \), since \( \beta^*_\alpha \) is the optimal value of model (3). This is a contradiction. \( \Box \)

Corollary 1. For each \( \beta \in [1, \beta^*_\alpha] \), the first level efficient units remain unchanged.

4. Illustrative examples

Charnes et al. (1981) used a multi-set DEA technique for data from a large-scale social experiment in public school education, Program Follow Through (PFT). The analysis compares the set of 49 schools receiving the intervention to a control group of 21 schools, denoted as Non-Follow Through (NFT). The study considers 5 inputs used in the production of three outputs. The inputs measure the education level of mother, the highest occupation level of a family member, the frequency of parental visits to the school, parental involvement with their children, and the number of teachers at a given site. The outputs measure reading and math ability as well as self-esteem.

Out of the 49 DMUs in set PFT, 17 are in \( E \) and 17 are in set \( E^* \). Here we employ the proposed model. The results are summarized in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>X4</th>
<th>X5</th>
<th>Y1</th>
<th>Y2</th>
<th>Y3</th>
</tr>
</thead>
<tbody>
<tr>
<td>PFT</td>
<td>0.76</td>
<td>0.16</td>
<td>0.05</td>
<td>0.01</td>
<td>0.06</td>
<td>0.13</td>
<td>0.33</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table 1: Result of new model for PFT data

To interpret the results of applying model (3), note that the second column of the table shows the perturbation of the first input, where the input of efficient DMUs can be increased 76% meanwhile the inefficient units can be decreased by 76% in such a way that the CCR efficient DMUs remain unchanged. Similar discussion is true for the remaining columns of Table 2. As another application we apply model (3) for the data given in Zhu et al. (1996) consisting of thirty Chinese textile
factories containing three inputs (circulating fund, investment, and labor) and three outputs (revenue from selling the products, profit and taxes, and net industrial output value).
Among the 30 DMUs under evaluation, 10 are in set $E$ and 8 are in set $E^*$. Again we use the proposed model (3). The results are given in Table 2.

<table>
<thead>
<tr>
<th>Textile</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$Y_1$</th>
<th>$Y_2$</th>
<th>$Y_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^*$</td>
<td>0.09</td>
<td>0.68</td>
<td>0.23</td>
<td>0.10</td>
<td>0.13</td>
<td>0.20</td>
</tr>
</tbody>
</table>

5. Conclusion

Since any data set might be affected by statistical noise, it is of high importance to determine the maximum perturbation in the data set. In this paper, by means of combining classic DEA models and preserving efficient DMUs, we determined the maximum perturbations of the data before efficient DMUs become inefficient. As the future research the same study can be discussed on the other DEA models (BCC, additive, etc) as well as discussing for a given group of inputs and outputs.

References


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