A Note on Random Solution for
Some Random Operator Equation

CHEN Ning
Department of Mathematics and Physics Southwest University of Science and Technology Mianyang 621010 Sichuan, P.R. China
cy783@yahoo.com.cn

CHEN Ji-qian
Department of Mathematics and Physics Southwest University of Science and Technology Mianyang 621010 Sichuan, P.R. China

Abstract
In this paper, some new results are also given for the random solution of a class of random operator equations which generalize several results in [4] (2004) and some notes in [5] in Banach space. The famous Altman’s inequality is also considered into the type of determinant form. The main results are theorem 3, theorem 5 and theorem 7. The example 1-2 expresses the application of theorem 7.

Keywords: Random operator equation, Random solution

1 Introduction and lemma
In this paper, we shall consider that the random solution of random operator equation by random fixed point theorem and the Altman’s type inequality, the method is similar as in [4] and [5], which generalize some results in [4] and [5].

\[\text{Keywords: Random operator equation, Random solution}\]

1 Introduction and lemma
In this paper, we shall consider that the random solution of random operator equation by random fixed point theorem and the Altman’s type inequality, the method is similar as in [4] and [5], which generalize some results in [4] and [5].

Lemma 1 (see theorem 1 in [4]) Assume that $E$ be a separable real Banach space, $X$ be a non-empty closed convex set in $E$, and $D$ be a bounded open set in $X$. Let $A : \Omega \times D \rightarrow X$ be random semi-closed 1-set contract operator, and $e_0 \in D$, such that

$$A(\omega, x) - \mu e_0 \neq \alpha (x - e_0)$$

where $\alpha$ be a variable that $\alpha > \mu \geq 1$, $\forall (\omega, x) \in \Omega \times \partial D$, then the $A(\omega, x) = \mu x$ must have a random solution in $D$.

2. Several theorems

Theorem 1 Assume that $E$ be a separable real Banach space, $X$ be a non-empty convex and closed set in $E$, $D$ be a bounded open set in $X$.

Let $A, B : \Omega \times \overline{D} \rightarrow X$ be a random semi-closed 1-set contract operator, and $\theta \in D$, real number $\mu \geq 1$, such that

$$\left\| B(\omega, x) - \lambda \mu x \right\|^6 + \left\| A(\omega, x) \right\|^6 \leq \left\| A(\omega, x) - \mu x \right\|^6 + \left\| \mu x \right\|^6, \forall (\omega, x) \in \Omega \times \partial D \tag{1}$$

Then the systems $A(\omega, x) = \mu x, B(\omega, x) = \lambda \mu x$ must have a common random solution in $D(\lambda > 0)$.

Proof: By the theorem 2 in [4], we take only $m = 5, \beta = 0$. Then we give that system: $A(\omega, x) = \mu x, B(\omega, x) = \lambda \mu x$ must have a common random solution in $D(\lambda > 0)$.

Theorem 2 Assume that as theorem, let $A, B, C : \Omega \times \overline{D} \rightarrow X$ are also random semi-closed 1-set contract operators, and $\theta \in D$, the real number $\mu \geq 1$, such that

$$\left\| C(\omega, x) - 3 \mu x \right\|^6 + \left\| B(\omega, x) - 2 \mu x \right\|^6 + \left\| A(\omega, x) \right\|^6 \leq \left\| A(\omega, x) - \mu x \right\|^6 + \left\| \mu x \right\|^6 \tag{2}$$

$\forall (\omega, x) \in \Omega \times \partial D$

Therefore the system

$$A(\omega, x) = \mu x, B(\omega, x) = 2 \mu x, C(\omega, x) = 3 \mu x$$

Must have a common random solution $x^*$ in $D$.

Proof: From theorem 2 in [4], we take only $m = 6, \beta = 0$ we get

$$\left\| A(\omega, x) \right\|^6 \leq \left\| A(\omega, x) - \mu x \right\|^6 + \left\| \mu x \right\|^6, \forall (\omega, x) \in \Omega \times \partial D$$

By lemma 1, this equation $A(\omega, x) = \mu x$ must have a random solution $x^*$ in $D$, that is $A(\omega, x^*) = \mu x^*$, then from (2), we have $\left\| B(\omega, x^*) - 2 \mu x^* \right\| \leq 0$. That is $B(\omega, x^*) = 2 \mu x^*$, then similar as above case, we must have that case: $C(\omega, x^*) = 3 \mu x^*$.

Therefore, the system: $A(\omega, x) = \mu x, B(\omega, x) = 2 \mu x, C(\omega, x) = 3 \mu x$ must have a
common random solution \( x^* \) in \( D \).

**Theorem 3** Suppose that same as theorem 1, and let \( A, B : \Omega \times D \rightarrow X \) are random semi-closed \( 1 \)-set contract operators ( \( i = 1, 2, \cdots, m \) ), and \( \theta \in D \), the real number \( \mu \geq 1 \), such that

\[
\sum_{i=1}^{m} \left\| B(\omega, x) - \lambda_i \mu x \right\|^\delta + \left\| A(\omega, x) - \mu x \right\|^\delta \leq \left\| A(\omega, x) - \mu x \right\|^\delta + \left\| \mu x \right\|^\delta, \forall (\omega, x) \in \Omega \times \partial D \quad (3)
\]

Then the system \( A(\omega, x) = \mu x, B_i(\omega, x) = \lambda_i \mu x, (i = 1, 2, \cdots, m) \) must have a common random solution in \( D(\lambda_i > 0) \).

**Proof** From theorem 2 in [4], we take only \( m = 6, \beta = 0 \), we get \( A(\omega, x) = \mu x \) must have a random solution \( x^* \) in \( D \), we substitute it in (3I by \( A(\omega, x^*) = \mu x^* \), then we have that

\[
\left\| B_i(\omega, x^*) - \lambda_i \mu x^* \right\|^\delta \leq 0
\]

Hence \( B_i(\omega, x^*) = \lambda_i \mu x^* \), (\( i = 1, 2, \cdots, m \))

Therefore, the system \( B_i(\omega, x^*) = \lambda_i \mu x^* \), must have a common random solution \( x^* \) in \( D \). We end the proof of this theorem.

**Corollary** when, \( \lambda_1 = \lambda_2 = \cdots = \lambda_m = \mu \) and \( B_i(\omega, x) = \mu^2 x \), or \( \lambda_1 = \lambda_2 = \cdots = \lambda_m = \mu^{-1} \), we get the special case.

**Theorem 4** Assume that similar as theorem 2, substituting (2) in the following form:

\[
\left\| C(\omega, x) - 3 \mu x \right\|^\delta + \left\| B(\omega, x) - 2 \mu x \right\|^\delta + \left\| A(\omega, x) - \mu x \right\|^\delta \leq \left\| A(\omega, x) - \mu x \right\|^\delta + \left\| \mu x \right\|^\delta, \forall (\omega, x) \in \Omega \times \partial D \quad (4)
\]

Then the system \( A(\omega, x) = \mu x, B(\omega, x) = 2 \mu x \) must have common solution in \( D \), or the \( A(\omega, x) = \mu x, C(\omega, x) = 3 \mu x \) must have common random solution in \( D \).

**Proof** From theorem 2 in [4], we take only \( m = 6, \beta = 0 \), (and similar as the proof of theorem 2, then we omit it).

3. **Altman type inequality**

**Theorem 5** Assume that \( X \) be a convex and closed set in separable real Banach space \( E \), \( D \) be a bounded open set in \( X \). Let \( A : \Omega \times \bar{D} \rightarrow X \) be a random semi-closed \( 1 \)-set contract operator, and \( \theta \in D \), real number \( \mu \geq 1 \), such that
\[ \|A(x) - \mu x\|^{2n} \geq D_{2n} = \begin{vmatrix} \|A(\omega, x)\|^2 & \|\mu x\|^2 & 0 & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \cdots & \|A(\omega, x)\|^2 & \|\mu x\|^2 \\ \|\mu x\|^2 & \cdots & 0 & \|A(\omega, x)\|^{2n} \end{vmatrix} \quad (5) \]

( where \( D_{2n} \) be determinant of \( 2n \)-order ) \( \forall (\omega, x) \in \Omega \times \partial D \) Then the equation \( A(\omega, x) = \mu x \) must have a random solution in \( D \).

**Proof** Let \( e_0 = \theta \in D \), By lemma 1, we only prove that \( A(\omega, x) \neq \alpha x \), \( \forall (\omega, x) \in \Omega \times \partial D \), \( \alpha > \mu \geq 1 \) and we assume the contrary is true. If there exists \( \alpha_0 > \mu \geq 1 \), \( \omega_0 \in \Omega, x_0 \in \partial D \) such that \( A(\omega_0, x_0) = \alpha_0 x_0 \). Similar as above stating, (by 5), we have \( \alpha_0 > \mu \geq 1, \omega_0 \in \Omega, x_0 \in \partial D \)

Let \( \alpha_0 - \mu = a \), then it reduces that \( a^{2n} + \mu^{2n} \geq (a + \mu)^{2n} \)
Which is a contradiction. By lemma 1, we complete the proof of this theorem.

**Theorem 6** Suppose that same as theorem 5, substituting (5) in the following form:
\[ \begin{vmatrix} \|A(\omega, x)\|^2 & \|\mu x\|^2 & 0 & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \cdots & \|A(\omega, x)\|^2 & \|\mu x\|^2 \\ \|\mu x\|^2 & \cdots & 0 & \|A(\omega, x)\|^{2n} \end{vmatrix} = D_n. \]

(1 - \( \epsilon \)) \( D_n \leq (1 + \epsilon) \|A(\omega, x) - \mu x\|^2 \cdot \|A(\omega, x) + \mu x\|^2 \cdot \forall (\omega, x) \in \Omega \times \partial D \) (6) then \( A(\omega, x) = \mu x \) must have a random solution in \( D(0 < \epsilon_1 < 1, 0 < \epsilon_2 < 1) \).

**Proof** Similar as the proof of theorem 5, by \( A(\omega_0, x_0) = \alpha_0 x_0 \), substituting it in (6), we have
\( (1 + \epsilon)(\alpha_0 - \mu)^n (\alpha_0 + \mu)^n \geq [\alpha_0^2 + (n-1)\mu^2] \cdot (\alpha_0^2 - \mu^2)^{n-1} (1 - \epsilon) \)
\( (\epsilon_1 + \epsilon_2)\alpha_0^2 - [(n-\epsilon_2) + (n-1)\epsilon_1] \geq 0. \)
And the discriminant is that \( \Delta = 4(\epsilon_1 + \epsilon_2)[n(1+\epsilon_1) + \epsilon_2 - \epsilon_1] > 0 \), it is a contradiction. then \( A(\omega, x) = \mu x \) must have a random solution in \( D \).

**Remark** : When \( \epsilon_1 = 0, \epsilon_2 = 0 \), \( n\mu^2 \leq 0 \), which is also contradiction, and it improve this results in [4].

**4. Some notes in [5]**

**Theorem 7** Suppose that same as theorem 5, substituting (5) in the following
form:

\[ (1 - \varepsilon)\|A(\omega, x)\|^2 \leq (1 + \varepsilon)\|\mu x + \delta A(\omega, x)\| \|\mu x - \delta A(\omega, x)\|, \quad \forall (\omega, x) \in \Omega \times \partial D \] (7)

Then \( A(\omega, x) = \mu x \) most have a random solution in \( D(0 \leq \delta \leq 1, \ 0 < \varepsilon < 1) \).

**Proof** Similar as the proof of theorem 5, by \( A(\omega_0, x_0) = \alpha_0 x_0 \) substituting it in (7) we get

\[ (1 - \varepsilon)\alpha_0^2 \leq (1 + \varepsilon)(\mu^2 - (\delta \alpha_0)^2) \]

That is, \((1 + \varepsilon)\mu^2 - [(1 + \varepsilon) + (1 + \varepsilon)\delta^2] \alpha_0^2 \geq 0 \), Which discriminant is

\[ \Delta = 4[(1 + \varepsilon)^2 \delta^2 + (1 - \varepsilon^2)] \alpha_0^2 > 0 \], that is a contradiction. This ends of the proof.

**Theorem 8** Suppose that same as theorem 8, substituting (8) in following form:

\[ \|A(\omega, x)\| \|2\mu x + \delta A(\omega, x)\| \geq \|\mu x - \delta A(\omega, x)\| \|\mu x + \delta A(\omega, x)\| \quad \forall (\omega, x) \in \Omega \times \partial D \] (8)

Then \( A(\omega, x) = \mu x \) must have a random solution in \( D \).

**Proof** Similar as the proof of theorem 7, by \( A(\omega_0, x_0) = \alpha_0 x_0 \) substituting it in (8) we get \( \alpha(2\mu - \delta \alpha_0) \geq \mu^2 - \delta^2 \alpha_0^2 \), that is \( \delta(1 + \delta) \alpha_0^2 + 2\mu \alpha_0 \mu^2 \geq 0 \), Which have that \( \Delta = 4(1 + \delta)^2 \delta^2 + 4\mu^2 > 0 \), we get the contradiction. This ends of the proof.

**Theorem 9** Suppose that same as theorem 8, substituting (9) in the following form:

\[ \|A(\omega, x)\|^2 \leq \|\mu x + \delta A(\omega, x)\| \|\mu x - \delta A(\omega, x)\| + 2 \|\mu x - \delta A(\omega, x)\|^2 \quad \forall (\omega, x) \in \Omega \times \partial D \] (9)

Then \( A(\omega, x) = \mu x \) most have a random solution in \( D \) (0 \leq \delta \leq 1) .

**Proof:** By lemma 1 we take \( e_0 = \theta \in D \), if there exists \( \alpha_0 > \mu \geq 1, \ \omega_0 \in \Omega, \ x_0 \in \partial D \) such that \( A(\omega_0, x_0) = \alpha_0 x_0 \), by (9) we have

\[ \alpha_0^2 \leq [\mu^2 - \delta^2 \alpha_0^2] + (\mu - \delta \alpha_0)^2 \]

We get \( 2\mu^2 - 2\mu \delta \alpha_0 - \alpha_0^2 \geq 0 \), which is that \( \Delta = (2\delta \alpha_0)^2 + 8\alpha_0^2 > 0 \), We get the contradiction , therefore we prove this theorem .

**5. Examples**

From the theorem 1 in [5] and the theorem 7 in this paper , we easy get following two examples ( similar as example 2 in [5] )

**Example 1.** We consider that non-linear random equation:

\[
\sin(x + 3\omega) + \frac{1}{2}\sin(x + \omega) - 2x = 0, \quad \omega \in [0, 1] = \Omega.
\]

Then this equation must have random solution in \([-\pi, \pi]\).
In fact, we is easy to prove that $A(\omega, x) = \frac{1}{2} \sin(x + 3\omega) + \frac{1}{4} \sin(x + \omega)$, (where $\omega \in [0, 1] = \Omega, x \in [-\pi, \pi]$) is a random semi-closed 1-set contract operator by seeing [5]), and by theorem 7 we take $\mu = 1, \delta = 0$, and at the boundary point in interval holds, $\|A(\omega, -\pi)\|^2 < \|\pi\|^2 = \pi^2$, $\|A(\omega, \pi)\|^2 < \|\pi\|^2 = \pi^2$, it satisfying the conditions (7), therefore we get that $A(\omega, x) = x$ must have random solution on $[-\pi, \pi]$, that is this non-linear equation

$$\sin(x + 3\omega) + \frac{1}{2} \sin(x + \omega) - 2x = 0, \omega \in [0, 1] = \Omega$$

which must have the random solution in $[-\pi, \pi]$.

**Example 2.** We consider the following equation similar as example 1:

$$\sin(x + 3\omega) + \frac{1}{4} \sin(x + \omega) - 3x = 0, \omega \in [0, 1] = \Omega$$

Which have random solution in $[-\pi, \pi]$.

6. Remark on that things

we may reference [1] and [7] for random differential equation (or with Random Impulsive. in [6])

References


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