Application of Variational Iteration Method to Linear Partial Differential Equations

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Abstract

The variational iteration method (VIM) is applied to the numerical simulation of linear third-order dispersive partial differential equations (PDEs). The VIM produces an approximate solution of the equation without any discretization. The VIM is based on the incorporation of a general Lagrange multiplier in the construction of correction functional for the equation. Several equations are chosen as test cases to illustrate the efficiency of the method. In VIM, a correction functional is constructed by a general Lagrange multiplier which can be identified via a variational theory. The VIM yields an approximate solution in the form of a quickly convergent series. Comparisons with exact solution show that the VIM is a powerful method for the solution of linear equations.

Mathematics Subject Classification: 65L05, 65L06, 35A15

Keywords: dispersive PDEs, variational iteration method, Lagrange multiplier

1 Introduction

It is well known that many phenomena in scientific fields can be described by partial differential equations [1, 2, 3, 4, 5]. Finding exact or approximate solutions of these equations is interesting and important. Both the heat conduction and wave equations have been examined extensively, analytically and numerically, by many authors. While the third-order dispersive equations have
not been studied to the same extent [6]. Dispersive PDE’s are equations relating time derivatives of a function to odd ordered derivatives with respect to the spatial variable. Relatively few such equations can be solved explicitly. A family of numerical methods has been developed by Djidjeli and Twizell [6] for the simulation of third-order dispersive equations in one space variable with time-dependent boundary conditions. Wazwaz [7] demonstrated that the exact solutions to some third-order dispersive PDEs can be derived using the Adomian decomposition method, cf. Adomian [8].

In recent years, a great deal of attention has been devoted to the application of the VIM for numerical simulations of a wide range of problems, see for example, [9, 10, 11, 12, 13, 14, 15]. In [15], Soliman effectively applied the VIM to solve the generalized regularized long wave equation. In this paper, we shall apply the VIM to the numerical simulations of third-order dispersive PDEs. Comparisons with the exact solutions will be made.

2 Variational iteration method

The VIM, first proposed by He [11], is a modified general Lagrange’s multiplier method [16]. The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as initial approximation or trial function, then a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to accurate solution, [11].

To illustrate the basic concepts of the VIM, we consider the following nonlinear equation,

\[ Lu + Nu = g(x,t), \] (1)

where \( L \) is a linear operator, \( N \) a nonlinear operator and \( g(x,t) \) is an inhomogeneous term. According to the VIM, [11, 12, 13], we can construct a correction functional as follows,

\[ u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)[Lu_n(x,s) + Nu_n(x,s) - g(x,s)] ds, \quad (n \geq 0) \] (2)

where \( \lambda \) is a general Lagrangian multiplier [16], which can be identified optimally via the variational theory, the subscript \( n \) denotes the \( n \)-th order approximation and \( \tilde{u}_n \) is considered as a restricted variation [11, 12, 13], i.e. \( \delta \tilde{u}_n = 0 \).
3 Applications

In this section we shall apply the VIM to linear third-order dispersive PDEs in one and higher dimensions.

3.1 One-dimensional dispersive equation

First, we consider a linear third-order dispersive PDEs in one dimension as given by

\[ u_t + au_x + bu_{xxx} = g(x, t), \quad L_0 < x < L_1, \quad t > 0, \quad a, b > 0, \]  

(3)

where \( g \) is a source term. Equation (3) is subject to the initial condition,

\[ u(x, 0) = f(x), \]  

(4)

and the boundary conditions,

\[ u(0, t) = f_0(t), \quad u_x(0, t) = f_1(t), \quad u_{xx}(0, t) = f_2(t), \quad t > 0. \]  

(5)

By the VIM, we construct a correction functional,

\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(s)\left[(u_n)_s + a(\tilde{u}_n)_x + b(\tilde{u}_n)_{xxx} - g\right] ds, \]  

(6)

where \( \tilde{u}_n \) is considered as restricted variations, i.e. \( \delta\tilde{u}_n = 0 \). Making (6) stationary, i.e.

\[ \delta u_{n+1}(x, t) = \delta u_n(x, t) + \delta \int_0^t \lambda(s)\left[(u_n)_s + a(\tilde{u}_n)_x + b(\tilde{u}_n)_{xxx} - g\right] ds, \]  

(7)

\[ \delta u_{n+1}(x, t) = \delta u_n(x, t) + \delta \int_0^t \lambda(s)(u_n)_s ds, \]  

(8)

\[ \delta u_{n+1}(x, t) = \delta u_n(x, t) + \lambda(s)\delta u_n(x, s) - \int_0^t \delta u_n(x, s)\lambda'(s) ds, \]  

(9)

which then yields the conditions,

\[ \lambda'(s) = 0 \text{ and } 1 + \lambda(s)|_{s=t} = 0. \]  

(10)

The Lagrange multipliers can therefore be identified as \( \lambda = -1 \), and hence the variational iteration formula for the simulation of problem (3)–(5) is given by

\[ u_{n+1}(x, t) = u_n(x, t) - \int_0^t [(u_n)_s + a(u_n)_x + b(u_n)_{xxx} - g] ds. \]  

(11)
For simplicity, we can take an initial approximation \( u_0 = u(x, 0) = f(x) \) as given by (4). The next two iterates are easily obtained from (11) and are given by

\[
\begin{align*}
\mathbf{u}_1(x, t) &= u_0(x, t) - \int_0^t [(u_0)_s + a(u_0)_x + b(u_0)_{xxx} - g] \, ds, \\
\mathbf{u}_2(x, t) &= u_1(x, t) - \int_0^t [(u_1)_s + a(u_1)_x + b(u_1)_{xxx} - g] \, ds.
\end{align*}
\]

(12)

(13)

In the same manner, the rest of the iterates can be obtained efficiently using a computer algebra package like Maple.

### 3.2 Higher-dimensional dispersive equation

Next, we consider a linear third-order dispersive PDE in three dimensional spaces given by

\[
u_t + au_{xxx} + bu_{yyy} + cu_{zzz} = g(x, y, z, t),
\]

(14)

where \( L_0 < x, y, z < L_1, t > 0, a, b, c > 0 \) and \( g \) is a source term.

The initial condition is

\[
u(x, y, z, 0) = f(x, y, z),
\]

(15)

and the time-dependent boundary conditions are assumed to be prescribed.

Again, formally in the VIM, the approximate solution of equation (14) with condition (15) can be obtained by first constructing a correction functional,

\[
u_{n+1}(x, y, z, t) = \nu_n(x, y, z, t) + \int_0^t \lambda(s)[(\nu_n)_s + a(\tilde{\nu}_n)_{xxx} + b(\tilde{\nu}_n)_{yyy} + c(\tilde{\nu}_n)_{zzz} - g] \, ds. \quad (n \geq 0)
\]

(16)

Making this stationary gives the same conditions as in (10) and hence the Lagrange multiplier is also \( \lambda = -1 \). So, the iteration formula (16) now becomes

\[
u_{n+1}(x, y, z, t) = \nu_n(x, y, z, t) - \int_0^t [(\nu_n)_s + a(\nu_n)_{xxx} + b(\nu_n)_{yyy} + c(\nu_n)_{zzz} - g] \, ds. \quad (n \geq 0)
\]

(17)

Again, starting from the initial approximation \( u_0 = u(x, y, z, 0) = f(x, y, z) \) as in (15), the next iterates are given by

\[
u_1(x, y, z, t) = u_0(x, y, z, t) - \int_0^t [(u_0)_s + a(u_0)_{xxx} + b(u_0)_{yyy} + c(u_0)_{zzz} - g] \, ds,
\]

(12)
Variational iteration method for LPDEs

\[ u_2(x, y, z, t) = u_1(x, y, z, t) - \int_0^t [(u_1)_s + a(u_1)_{xxx} + b(u_1)_{yyy} + c(u_1)_{zzz} - g] \, ds, \]  

(18)
eq \int_0^t [u_1(s) + a(u_1)_s + b(u_1)_{xxx} + c(u_1)_{yyy} - g] \, ds,  

(19)
eq \int_0^t [(u_1)_s + a(u_1)_{xxx} + b(u_1)_{yyy} + c(u_1)_{zzz} - g] \, ds,  

eq \int_0^t [(u_1)_s + a(u_1)_{xxx} + b(u_1)_{yyy} + c(u_1)_{zzz} - g] \, ds,  

etc.

4 Test problems

In this section, we shall determine the accuracy of the VIM in solving some linear third-order dispersive PDEs. Comparisons will made against the exact solutions.

The test problems we shall consider are the following:

Problem 1 Homogeneous:

\[ u_t + 2u_x + u_{xxx} = 0, \quad t > 0. \]  
\[ u(x, 0) = \sin x, \]  
\[ u_{\text{exact}} = \sin(x - t). \]  

(20)

(21)

(22)

Problem 2 Nonhomogeneous:

\[ u_t + u_{xxx} = -\sin(\pi x) \sin t - \pi^3 \cos(\pi x) \cos t, \quad 0 < x < 1, \quad t > 0, \]  
\[ u(x, 0) = \sin(\pi x), \]  
\[ u_{\text{exact}} = \sin(\pi x) \cos t. \]  

(23)

(24)

(25)

Problem 3 Two-dimensional:

\[ u_t + u_{xxx} + u_{yyy} = 0, \quad t > 0, \]  
\[ u(x, y, 0) = \cos(x + y), \]  
\[ u_{\text{exact}} = \cos(x + y + 2t). \]  

(26)

(27)

(28)

Problem 4 Three-dimensional:

\[ u_t + u_{xxx} + \frac{1}{8} u_{yyy} + \frac{1}{27} u_{zzz} = -3 \cos(x + 2y + 3z) \sin t \]  
\[ + \sin(x + 2y + 3z) \cos t, \quad t > 0, \]  
\[ u(x, y, z, 0) = 0, \]  
\[ u_{\text{exact}} = \sin(x + 2y + 3z) \sin t. \]  

(29)

(30)

(31)
Table 1: Absolute errors between the 5th-iterate of VIM, $u_5$, and the exact solutions for Problems 1 and 2.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$t$</th>
<th>$u_5$ (Problem 1)</th>
<th>$u_5$ (Problem 2)</th>
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<td>0.2</td>
<td>6.290E-9</td>
<td>4.783E-03</td>
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<td>0.2</td>
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<td>0.9</td>
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<td>0.4</td>
<td>4.242E-6</td>
<td>8.884E-02</td>
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</table>

Using (11) we obtain the 5th-iterates

$$u_5(x, t) = \sin(x) - \cos(x)t - \frac{1}{2} \sin(x)t^2 + \frac{1}{6} \cos(x)t^3 + \frac{1}{24} \sin(x)t^4 - \frac{1}{120} \cos(x)t^5,$$

(32)

$$u_5(x, t) = \sin(\pi x) \cos t + 2\pi^3 \cos(\pi x) \sin t - 2\sin(\pi x)\pi^6 + 2\pi^6 \sin(\pi x) \cos t - 2\cos(\pi x)\pi^{12} t - 2\pi^9 \cos(\pi x) \sin t - 2\sin(\pi x)\pi^{12} t$$

$$+ \sin(\pi x)\pi^{12} t^2 + 2\pi^{12} \sin(\pi x) \cos t - \cos(\pi x)\pi^{15} t$$

$$+ \frac{1}{6} \cos(\pi x)\pi^{15} t^3 + \frac{1}{120} \cos(\pi x)\pi^{15} t^5 + \pi^{15} \cos(\pi x) \sin t,$$

(33)

for Problems 1 and 2, respectively. In Table 1 we present the absolute errors for Problems 1 and 2.

Using (17) we obtain the 5th-iterates

$$u_5(x, t) = \cos(x + y) - 2\sin(x + y)t - 2\cos(x + y)t^2$$

$$+ \frac{4}{3} \sin(x + y)t^3 + \frac{2}{3} \cos(x + y)t^4 - \frac{4}{15} \sin(x + y)t^5,$$

(34)

$$u_5(x, t) = -\frac{438443}{4096} \cos(x + 2y + 3z) + \frac{438843}{4096} \cos(x + 2y + 3z) \cos t$$

$$+ \frac{500361}{4096} \sin(x + 2y + 3z) \sin t - \frac{496265}{4096} \sin(x + 2y + 3z) t$$

$$+ \frac{4394075}{8192} \cos(x + 2y + 3z) t^2 + \frac{471625}{24576} \sin(x + 2y + 3z) t^3$$

$$- \frac{1500625}{32768} \cos(x + 2y + 3z) t^4,$$

(35)

for Problems 3 and 4, respectively. In Table 2 we present the absolute errors for Problems 3 and 4.
Table 2: Absolute errors between the 5th-iterate of VIM, $u_5$, and the exact solutions for Problems 3 and 4.

<table>
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<th>$t$</th>
<th>$u_5$</th>
<th>$z$</th>
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<td>3.524E-02</td>
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<td>1.503E-01</td>
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</tbody>
</table>

5 Conclusions

In this paper, the VIM has been successfully applied to the simulation of third-order dispersive PDEs. The results show that the VIM is a simple and reliable method for finding approximate solutions to linear PDEs. All the algebraic manipulations in this work have been carried out using the Maple algebra package.

Acknowledgment

B. Batiha would like to acknowledge the financial support received from Philadelphia University.

References


Received: March, 2009