Triple Positive Solutions of Four-Point Singular Boundary Value Problems for $p$-Laplacian Dynamic Equations on Time Scales

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Abstract

Let $T$ be a time scale. We study the existence of three positive solutions for the nonlinear four-point singular boundary value problem with $p$-Laplacian operator on time scales. Some new results are obtained for three positive solutions by applying Leggett-Williams fixed-point theorem in a cone, which is different from the previous.

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1 Introduction

A time scale $T$ is a nonempty closed subset of $\mathbb{R}$. We make the blanket assumption that $0, T$ are point in $T$. By an internal $(0, T)$, we always mean the intersection of the real internal $(0, T)$ with the given time scale, that is $(0, T) \cap T$. The theory of dynamical systems on time scales is undergoing rapid development as it provides a unifying structure for the study of differential equations in the continuous case and the study of finite difference equations in the discrete case; here, two-point boundary-value problems have been extensively studied; see [1, 2, 3, 4, 5] and the references therein. However, there are not much concerning the $p$-Laplacian problems on time scales.

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In this paper, we study the existence of three positive solutions for the following nonlinear four-point singular boundary value problem with $p$-Laplacian operator on time scales:

\[
\begin{cases}
    (\phi_p(u^\Delta))^{\nabla} + a(t)f(u(t)) = 0, \quad t \in (0, T), \\
    \alpha \phi_p(u(0)) - \beta \phi_p(u^\Delta(\xi)) = 0, \quad \gamma \phi_p(u(T)) + \delta \phi_p(u^\Delta(\eta)) = 0,
\end{cases}
\]

(1.1)

where \( \phi_p(s) \) is a \( p \)-Laplacian operator, i.e. \( \phi_p(s) = |s|^{p-2}s, \quad p > 1, \quad \phi_q = \phi_p^{-1}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \xi, \eta \in (0, T) \) is prescribed and \( \xi < \eta, \quad a : (0, 1) \to [0, \infty), \quad \alpha > 0, \quad \beta \geq 0, \quad \gamma > 0, \quad \delta \geq 0, \) and

\[ (H_1) \quad f \in C([0, +\infty), [0, +\infty)); \]

\[ (H_2) \quad a : (0, T) \to [0, +\infty) \text{ is left dense continuous (i.e.,} a(t) \in C_{ld}((0, T), [0, +\infty)) \text{ and there exists} \quad t_0 \in (0, T), \text{ such that} \]

\[ a(t_0) > 0, \quad 0 < \int_0^T a(s) \nabla s < +\infty. \]

For convenience, we list here the following definitions which are needed later.

A time scale \( T \) is an arbitrary nonempty closed subset of real numbers \( R \). The operators \( \sigma \) and \( \rho \) from \( T \) to \( T \) which defined by [1-5],

\[ \sigma(t) = \inf\{\tau \in T \mid \tau > t\} \in T, \quad \rho(t) = \sup\{\tau \in T \mid \tau < t\} \in T. \]

are called the forward jump operator and the backward jump operator, respectively.

The point \( t \in T \) is left-dense, left-scattered, right-dense, right-scattered if \( \rho(t) = t, \rho(t) < t, \sigma(t) = t, \sigma(t) > t, \) respectively. If \( T \) has a right scattered minimum \( m, \) define \( T_k = T - \{m\}; \) otherwise set \( T_k = T. \) If \( T \) has a left scattered maximum \( M, \) define \( T^k = T - \{M\}; \) otherwise set \( T^k = T. \)

Let \( f : T \to R \) and \( t \in T^k \) (assume \( t \) is not left-scattered if \( t = \sup T \) ),

then the delta derivative of \( f \) at the point \( t \) is defined to be the number \( f^\Delta(t) \) (provided it exists) with the property that for each \( \epsilon > 0 \) there is a neighborhood \( U \) of \( t \) such that

\[ |f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq |\sigma(t) - s|, \quad \text{for all} \quad s \in U. \]

Similarly, for \( t \in T \) (assume \( t \) is not right-scattered if \( t = \inf T \) ),

the nabla derivative of \( f \) at the point \( t \) is defined in [1] to be the number \( f^\nabla(t) \) (provided it exists) with the property that for each \( \epsilon > 0 \) there is a neighborhood \( U \) of \( t \) such that

\[ |f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq |\rho(t) - s|, \quad \text{for all} \quad s \in U. \]

If \( T = R, \) then \( x^\Delta(t) = x^\nabla(t) = x(t). \) If \( T = Z, \) then \( x^\Delta(t) = x(t + 1) - x(t) \)
is the forward difference operator while \( x^\nabla(t) = x(t) - x(t - 1) \) is the backward difference operator.
2 Preliminaries and Lemmas

In this section, we give some definitions and preliminaries.

**Definition 2.1.** Let $E$ be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that

(i) $u \in P$, $a \geq 0$ implies $au \in P$; and

(ii) $u, -u \in P$ implies $u = 0$.

**Definition 2.2.** Given a cone $P$ in a real Banach space $E$, a functional $\alpha : P \to [0, \infty)$ is said to be nonnegative continuous concave on $P$ such that

$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$, for all $x, y \in P$ with $t \in [0, 1]$.

Let $a, b, r > 0$ be constants with $P$ and $\alpha$ as defined above, we note

$P_r = \{ y \in P | \|y\| < r \}$, $P\{\alpha, a, b\} = \{ y \in P | \alpha(y) \geq a, \|y\| \leq b \}$.

Our main tool of this paper is the following the fixed point theorem due to Leggett-Williams (see[6, 7]).

**Theorem 2.1 (Leggett-Williams).** Assume $E$ be a real Banach space, $P \subset E$ be a cone. Let $A : P \to P$ be completely continuous and $\alpha$ be a nonnegative continuous concave functional on $P$ such that $\alpha(y) \leq \|y\|$, for $y \in P$.

Suppose that there exist $0 < a < b < d \leq c$ such that

(C1) $\{ y \in P(\alpha, b, d) | \alpha(y) > b \} \neq \emptyset$ and $\alpha(Ay) > b$, for all $y \in P(\alpha, b, d)$;

(C2) $\|Ay\| < a$, for all $\|y\| \leq a$;

(C3) $\alpha(Ay) > b$ for all $y \in P(\alpha, b, c)$ with $\|Ay\| > d$.

Then $A$ has at least three fixed points $y_1, y_2, y_3$ satisfying

$\|y_1\| < a, \quad b < \alpha(y_2)$,

and

$\|y_3\| > a, \quad \alpha(y_3) < b$.

Let $E = C_d([0, T], R)$ which is a Banach space with the maximum norm $\|u\| = \max_{t \in [0, T]} |u(t)|$. And define the cone $P \subset E$ by

$P = \{ u \in E : u(t) \geq 0, \quad u(t) \text{ is concave function, } t \in [0, T] \}$.

We can easily get the following lemmas.

**Lemma 2.1.** Suppose condition $(H_2)$ holds, then there exists a constant $\mu \in (0, \frac{1}{2})$ satisfies

$0 < \int_{\mu}^{T-\mu} a(t) \nabla t < \infty$. 
Furthermore, the function
\[ y_1(t) = \phi_q \left( \int_{\mu}^t a(t) \nabla t \right) + \phi_q \left( \int_{t}^{T-\mu} a(t) \nabla t \right), \quad t \in [\mu, T-\mu]. \]
is positive continuous functions on \([\mu, T-\mu]\), therefore \(y_1(t)\) has minimum on \([\mu, T-\mu]\), hence we suppose that there exists \(L_1 > 0\) such that \(L_1 = \min_{t \in [\mu, T-\mu]} y_1(t)\).

**Lemma 2.2.** Let \(u \in P\) and \(\mu\) of Lemma 2.1, then
\[ u(t) \geq \mu \|u\|, \quad t \in [\mu, T-\mu]. \]

The proof of Lemma 2.2 is similar to the proof of Lemma 2.2 in [9], so we omit the details.

Now, we define a mapping \(A : P \rightarrow E\) given by
\[
(Au)(t) = \begin{cases} 
\phi_q \left( \int_{0}^{\sigma} a(r)f(u(r))\nabla r \right) + \int_{0}^{\tau} \phi_q \left( \int_{s}^{\sigma} a(r)f(u(r))\nabla r \right) \Delta s, & 0 \leq t \leq \sigma, \\
\phi_q \left( \int_{\sigma}^{T} a(r)f(u(r))\nabla r \right) + \int_{\sigma}^{T} \phi_q \left( \int_{s}^{T} a(r)f(u(r))\nabla r \right) \Delta s, & \sigma \leq t \leq T.
\end{cases}
\]
Because of
\[
(Au)^{\Delta}(t) = \begin{cases} 
\phi_q \left( \int_{0}^{\sigma} a(r)f((u)(r))\nabla r \right) \geq 0, & 0 \leq t \leq \sigma, \\
-\phi_q \left( \int_{\sigma}^{T} a(r)f((u)(r))\nabla r \right) \leq 0, & \sigma \leq t \leq T,
\end{cases}
\]
the operator \(A\) is monotone decreasing continuous and \((Au)^{\Delta}(\sigma) = 0\), and for any \(u \in P\), we have
\[
(\phi_q(Au)^{\Delta})\nabla(t) = -a(t)f((u)(t)), \quad a.e. \ t \in (0, 1),
\]
and \((Au)(\sigma) = \|A(u)\|\). Therefore we have \(A(P) \subset P\).

**Lemma 2.3.** \(A : P \rightarrow P\) is completely continuous.

**Proof.** Suppose \(D \subset P\) is a bounded set, Let \(M > 0\) such that \(\|u\| \leq M\), \(u \in D\). For any \(u \in D\), we have
\[
\|Au\| = (Au)(\sigma) = \phi_q \left( \int_{0}^{\sigma} a(r)f(u(r))\nabla r \right) + \int_{0}^{\tau} \phi_q \left( \int_{s}^{\sigma} a(r)f(u(r))\nabla r \right) \Delta s,
\]
\[
\leq \left[ \phi_q \left( \int_{0}^{\tau} a(r)\nabla r \right) + \int_{0}^{\tau} \phi_q \left( \int_{s}^{\tau} a(r)\nabla r \right) \Delta s \right] \phi_q \left( \sup_{u \in D} f(u) \right).
\]
Then \(A(D)\) is bounded.

Furthermore it is easy see by Arzela-ascoli theorem and Lebesgue dominated convergence theorem that \(A : P \rightarrow P\) is completely continuous. The proof is complete.
3 Main Results

For notational convenience, we introduce the following constants:

\[ \lambda = \mu L_1, Q = (T + \phi_q \left( \frac{\beta}{\alpha} \right)) \phi_q \left( \int_0^T a(r) \nabla r \right). \]

Finally, we define the nonnegative, continuous concave functional \( \alpha : P \to [0, \infty) \) by

\[ \alpha(u) = \frac{1}{2} [u(\mu) + u(T - \mu)]. \]

We observe here that, for every \( u \in P \), \( \alpha(u) \leq \|u\| \).

Our main result of this paper is as follows:

**Theorem 3.1.** Suppose that conditions \((H_1), (H_2)\) hold. Let \( a < \mu b < b < \frac{b}{\mu} = d \leq c \) and assume that the following conditions are satisfied

\( (A_1) \): \( f(u) < \phi_p(a Q), \) for \( 0 \leq u \leq a; \)

\( (A_2) \): \( \) there exists a number \( q > d \) such that \( f(u) < \phi_p(q), \) for \( 0 \leq u \leq q; \)

\( (A_3) \): \( f(u) \geq \phi_p(2b), \) for \( \mu b \leq u \leq d. \)

Then, the boundary value problem \((1.1)\) has at least three positive solutions \( u_1, u_2 \) and \( u_3 \) such that

\[ \|u_1\| < a, \quad b < \alpha(u_2), \]

and

\[ \|u_3\| > a, \quad \alpha(u_3) < b. \]

**Proof.** By Lemma 2.3, we can know \( A : P \to P \) is completely continuous. Now we show that the conditions of Theorem 2.1 are satisfied. We first assert that if there exists a positive number \( c \) where \( c > d \) such that \( A(P_c) \subset P_c. \)

Suppose that condition \( (A_2) \) holds, if \( u \in P_q, \) then

\[ \|Au\| = (Au)(\sigma) \]

\[ = \phi_q \left( \frac{\alpha}{a} \int_0^a a(r)f((u)(r)) \nabla r \right) + \int_0^a \phi_q \left( f^T \phi_q \left( f_0^T a(r)f((u)(r)) \nabla r \right) \right) \Delta s \]

\[ \leq \phi_q \left( \frac{\alpha}{a} \int_0^a a(r)f((u)(r)) \nabla r \right) + \int_0^a \phi_q \left( f_0^T a(r)f((u)(r)) \nabla r \right) \Delta s \]

\[ = \left[ T + \phi_q \left( \frac{\beta}{\alpha} \right) \right] \phi_q \left( f_0^T a(r)f((u)(r)) \nabla r \right) \leq \left[ T + \phi_q \left( \frac{\beta}{\alpha} \right) \right] \phi_q \left( f_0^T a(r) \nabla r \right) \frac{q}{Q} \]

\[ = q. \]

Thus \( Au \in P_q. \) Therefore, taking \( c = q, \) we have \( A(P_c) \subset P_c. \) Especially, if \( u \in P_a, \) then assumption \( (A_1) \) implies \( f(u) < \phi_p(a Q), \) for \( 0 \leq u \leq a, \) from which we obtained above, we have \( A : P_a \to P_a. \)
To fulfill condition \((C_1)\) of Theorem 2.1, let \(u(t) \equiv \frac{b+d}{2}\), then \(u \in P, \|u\| = \frac{b+d}{2}\) and \(\alpha(u) = \frac{b+d}{2} > b\). That is \(\{u \in P(\alpha, b, d) | \alpha(u) > b\} \neq \emptyset\). Moreover, if \(u \in P(\alpha, b, d)\), then \(\alpha(u) \geq b\), so \(b \leq \|u\| \leq d\). By Lemma 2.2 we have \(\mu b \leq \mu \|u\| \leq u(t) \leq d\), for \(t \in [\mu, T - \mu]\). From assumption \((A_3)\) implies \(f(u) \geq \phi_p\left(\frac{2b}{\lambda}\right)\), for \(\mu b \leq u \leq d\).

We shall discuss it from three perspectives.

\((i)\) If \(\sigma \in [\mu, T - \mu]\), we have

\[
2\alpha(A(u)) = (Au)(\mu) + (Au)(T - \mu) \\
\geq \int_0^\mu \phi_q\left(\int_\sigma^\mu a(r)f((u)(r))\nabla r\right) \Delta s + \int_{T - \mu}^T \phi_q\left(\int_\sigma^T a(r)f((u)(r))\nabla r\right) \Delta s \\
\geq \mu \left(\phi_q\left(\int_\mu^\mu a(r)\nabla r\right) + \phi_q\left(\int_{T - \mu}^T a(r)\nabla r\right)\right) \frac{2b}{\lambda} \\
\geq b > b.
\]

\((ii)\) If \(\sigma \in (T - \mu, T]\), we have

\[
\alpha(A(u)) = \frac{1}{2}\left(\frac{(Au)(\mu) + (Au)(T - \mu)}{\mu}\right) \geq (A(u))(\mu) \\
\geq \int_0^\mu \phi_q\left(\int_\sigma^\mu a(r)f((u)(r))\nabla r\right) \Delta s \geq \int_{T - \mu}^T \phi_q\left(\int_\sigma^T a(r)f((u)(r))\nabla r\right) \Delta s \\
\geq \mu \phi_q\left(\int_{T - \mu}^T a(r)\nabla r\right) \frac{2b}{\lambda} \geq 2b > b.
\]

\((iii)\) If \(\sigma \in [0, \mu]\), we have

\[
\alpha(A(u)) = \frac{1}{2}\left(\frac{(Au)(\mu) + (Au)(T - \mu)}{\mu}\right) \geq (A(u))(T - \mu) \\
\geq \int_{T - \mu}^T \phi_q\left(\int_\sigma^{T - \mu} a(r)f((u)(r))\nabla r\right) \Delta s \geq \int_{T - \mu}^T \phi_q\left(\int_\sigma^{T - \mu} a(r)f((u)(r))\nabla r\right) \Delta s \\
\geq \mu \phi_q\left(\int_{T - \mu}^T a(r)\nabla r\right) \frac{2b}{\lambda} \geq 2b > b.
\]

Therefore, condition \((C_1)\) of Theorem 2.1 is satisfied.

Finally, we address condition \((C_3)\) of Theorem 2.1. For this we choose \(u \in P(\alpha, b, c)\) with \(\|Au\| > d\). Then from Lemma 2.2, we have

\[
\alpha(Au) = \frac{1}{2}\left(\frac{(Au)(\mu) + (Au)(T - \mu)}{\mu}\right) \geq \mu \|Au\| > \mu d = b.
\]

Hence, condition \((C_3)\) of Theorem 2.1 holds.

To sum up, all the hypotheses of Theorem 2.1 are satisfied. Hence \(A\) has at least three fixed points, i.e., the boundary value problem (1.1) has at least three positive solutions \(u_1, u_2\) and \(u_3\) such that

\[
\|u_1\| < a, \quad b < \alpha(u_2),
\]

and

\[
\|u_3\| > a, \quad \alpha(u_3) < b.
\]
Theorem 3.2. Suppose that conditions \((H_1), (H_2)\) hold. Let \(a < \mu b < b < \frac{b}{\mu} = d \leq c\) and assume that the following conditions are satisfied

\((B_1)\): \(f(u) < \phi_p\left(\frac{a}{Q}\right)\), for \(0 \leq u \leq a\);
\((B_2)\): \(\lim_{u \to \infty} \frac{f(u)}{\phi_p(u)} < \phi_p\left(\frac{1}{Q}\right)\);
\((B_3)\): \(f(u) \geq \phi_p\left(\frac{d}{Q}\right)\), for \(\mu b \leq u \leq d\).

Then, the boundary value problem (1.1) has at least three positive solutions \(u_1, u_2\) and \(u_3\) such that

\[ ||u_1|| < a, \quad b < \alpha(u_2), \]

and

\[ ||u_3|| > a, \quad \alpha(u_3) < b. \]

Proof. By Lemma 2.3 and Lemma 2.4, we can know  \(A : P \to P\) is completely continuous. Now we show that the conditions of Theorem 2.1 are satisfied. We only prove that there exists a positive number \(c\) where \(c > d\) such that  \(A(\overline{P}c) \subset \overline{P}c\) when condition \((B_2)\) holds. Suppose that condition \((B_2)\) holds, then there exist \(L > 0\) and \(\varepsilon < \phi_p\left(\frac{1}{Q}\right)\) such that \(f(u) < \phi_p\left(\frac{1}{Q}\right)\) for \(u > L\).

Set \(M = \{f(u) | 0 \leq u \leq L\}\), then

\[ f(u) \leq M + \varepsilon \phi_p(u) \quad \text{for} \quad u \geq 0. \]

We choose \(c\) such that

\[ \phi_p(c) > \max \left\{ \phi_p(d), \quad M \left( \phi_p\left(\frac{1}{Q} - \varepsilon\right)\right)^{-1} \right\}. \]

If \(u \in \overline{P}c\), then

\[ ||Au|| = (Au)(\sigma) \]
\[ = \phi_q\left(\frac{\alpha}{\alpha} f^\alpha a(r)f((u)(r))\nabla r\right) + \int_0^\sigma \phi_q\left(\int_s^\sigma a(r)f((u)(r))\nabla r\right) \Delta s \]
\[ \leq \phi_q\left(\frac{\alpha}{\alpha} f^T a(r)f((u)(r))\nabla r\right) + \int_0^T \phi_q\left(\int_0^T a(r)f((u)(r))\nabla r\right) \Delta s \]
\[ = \left[T + \phi_q\left(\frac{\alpha}{\alpha}\right)\right] \phi_q\left(\int_0^T a(r)f((u)(r))\nabla r\right) \]
\[ \leq \left[T + \phi_q\left(\frac{\alpha}{\alpha}\right)\right] \phi_q\left(\int_0^T a(r)\left(\phi_p(c) \times (\phi_p\left(\frac{1}{Q} - \varepsilon\right) + \varepsilon \phi_p(c)\right)\nabla r\right) \]
\[ \leq \left[T + \phi_q\left(\frac{\alpha}{\alpha}\right)\right] \phi_q\left(\int_0^T a(r)\nabla r\right) \frac{c}{Q} \]
\[ = c. \]
Thus $Au \in \overline{P}_c$. Therefore, we have $A(\overline{P}_c) \subset \overline{P}_c$. Especially, if $u \in \overline{P}_a$, then assumption $(A_1)$ implies $f(u) < \phi_p(\frac{a}{\mu})$, for $0 \leq u \leq a$, from which we obtained above, we have $A : \overline{P}_a \to P_a$. The rest proof of Theorem 3.2 is similar to Theorem 3.1, we omit it.

From Theorem 3.1, we see that, when assumption like $(A_1)$, $(A_2)$ and $(A_3)$ are imposed appropriately on $f$, we can establish the existence of an arbitrary odd number of positive solutions of boundary value problem (1.1).

**Theorem 3.3.** Suppose that conditions $(H_1)$, $(H_2)$ hold. Let $a_1 < \mu b_1 < b_1 < \frac{b_1}{\mu} = d_1 \leq a_2 < \mu b_2 < b_2 < \frac{b_2}{\mu} = d_2 \leq \cdots \leq a_n$, $n \in \mathbb{N}$ and assume that the following conditions are satisfied

$(D_1)$: $f(u) < \phi_p(\frac{a_i}{\mu})$, for $0 \leq u \leq a_i$, $i = 1, 2, \cdots, n$;  
$(D_2)$: $f(u) \geq \phi_p(\frac{2b_i}{\lambda})$, for $\mu b_i \leq u \leq d_i$, $i = 1, 2, \cdots, n$.

Then, the boundary value problem (1.1) has at least $2n - 1$ positive solutions.

**Proof.** When $n = 1$, it follows from condition $(D_1)$ that $A : \overline{P}_a \to P_a$. Therefore there exists at least one fixed point $u_1 \in \overline{P}_a$, by the Schauder fixed point theorem. When $n = 2$, it is that Theorem 3.1 holds (with $d_1 = \frac{b_1}{\mu}$, $c_1 = a_2$), so we can get at least three positive solutions $u_1$, $u_2$ and $u_3$ such that $||u_1|| < a$, $b < \alpha(u_2)$, and $||u_3|| > a$, $\alpha(u_3) < b$. Following the identical fashion, by the induction method we immediately complete the proof.

4 Application

In this section, we give an example as an application to illustrate our results.

**Example.** Consider the following singular boundary value problem (SBVP) with $p$-Laplacian

\[
\begin{align*}
\phi_p(u(\Delta)) + a(t)f(u) &= 0, \quad 0 < t < \frac{3}{2}, \\
4\phi_p(u(0)) - \phi_p(u(\frac{1}{4})) &= 0, \quad \phi_p(u(\frac{3}{2})) + \delta \phi_p(u(1)) &= 0,
\end{align*}
\]

(4.1)

where

\[
\beta = \gamma = 1, \quad \alpha = 4, \quad p = \frac{3}{2}, \quad \delta \geq 0, \quad \xi = \frac{1}{4}, \quad \eta = \frac{1}{2}, \quad T = \frac{3}{2}, \quad a(t) = \frac{1}{4}t^{-\frac{1}{2}},
\]

and

\[
f(u) = \begin{cases} 
1, & 0 \leq u \leq 2, \\
4u + 11, & 2 \leq u \leq 9, \\
47, & u \geq 9. 
\end{cases}
\]
Obviously, \( q = 3 \), we choose \( \mu = \frac{1}{4} \), then it is easy to see by calculating that

\[
L_1 = \min_{t \in [\mu, T-\mu]} y_1(t) = 3 - \sqrt{5},
\]

\[
Q = \frac{25\sqrt{6}}{64}.
\]

Let \( a = 2 \), \( b = 9 \), \( c = 6250 \). Then we have

\[
2 < \frac{9}{4} < 9 < 36 \leq 6250,
\]

\[
f(u) = 1 < \phi_p\left( \frac{a}{Q} \right) = \sqrt{\frac{2 \times 32\sqrt{6}}{75}}, \quad \text{for } 0 \leq u \leq 2,
\]

\[
f(u) \leq 47 < \phi_p\left( \frac{c}{Q} \right) = \sqrt{\frac{6250 \times 32\sqrt{6}}{75}}, \quad \text{for } 0 \leq u \leq 6250,
\]

\[
f(u) \geq 20 > \phi_p\left( \frac{2b}{\lambda} \right) = \sqrt{\frac{2 \times 9 \times 4}{3 - \sqrt{5}}}, \quad \text{for } \frac{9}{4} \leq u \leq 36.
\]

Therefore, by Theorem 3.1 we can know SBVP (4.1) has at least three positive solutions \( u_1, u_2 \) and \( u_3 \) satisfying

\[
||u_1|| < 2, \quad 9 < \alpha(u_2),
\]

and

\[
||u_3|| > 2, \quad \alpha(u_3) < 6.
\]

References


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