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# A Mathematical Approach to Solve Data Envelopment Analysis Models when Data are LR Fuzzy Numbers

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**Abstract**

data envelopment analysis is a linear programming methodologie that has been widely used to evaluate the performance of a set of decision-making units. It requires crisp input and output data. However, in reality input and output cannot be measured in a precise manner. We develop DEA models using imprecise data represented by LR fuzzy numbers with different shapes. The resulting FDEA models take the form of fuzzy linear programming and can be solved by the use of some approaches to rank fuzzy numbers. As an alternative, we introduce an approach based on the ordering relations between LR-fuzzy numbers. The approach transforms FDEA models into crisp linear programming problems. We used a numerical example to illustrate the approach and compare the results with other approaches.

**Keywords:** Data envelopment analysis, fuzzy mathematical programming, LR fuzzy numbers, efficiency analysis, ordering relations

# Introduction

Data Envelopment Analysis (DEA), developed by Charnes et al. [1] has emerged as an important tool to evaluate the efficiency of a set of "Decision Making Units" (DMUs) using multiple inputs to product multiple outputs. It has been extensively applied in performance evaluation and benchmarking in a wide variety of contexts including educational departments in public schools and universities, health care units, prisons, agricultural production, and banks.

DEA determines a set of weights such that the efficiency of DMU0 (target DMU) relative to the other DMUS is maximised. For an inefficient DMU, it identifies the source and the amount of inefficiency in each input relative to each output. While traditional DEA requires precise data for its analysis, the evaluation environment often involves vagueness and uncertainty. As system complexity increases, obtaining precise data becomes a difficult task. Furthermore, decision-makers often think and operate based on vague linguistic data (e.g., quality is "good", on time performance is "poor"). In these cases, fuzzy set theory can be a powerful tool to quantify imprecise and vague data in DEA models. FDEA models (DEA models with fuzzy inputs and fuzzy outputs) take the form of fuzzy linear programming models. The resolution of fuzzy linear programming requires a technique to rank fuzzy sets. Many researchers dealing with the problem of fuzzy sets can be founded in [11, 12, and 13]. In this paper, we propose an approach based on the ordering relations between LR-fuzzy numbers to solve the primal and the dual of FCCR. The inputs and outputs are represented by power fuzzy numbers with different shapes. In this approach we suggest a procedure based on the resolution of a goal programming problem to transform the fuzzy normalisation equality in the primal of FCCR. In this way, our approach could be a really useful methodology that provides practitioners with models which represent some real live processes more appropriately. The remaining part of this paper is organized as follows. In section 2, DEA and FDEA models are represented. Section 3 is devoted to data modelling with LR-fuzzy numbers. In section 4, we use the proposed approach to solve FDEA models. Two numerical examples are presented. Finally, section 5 concludes the paper.

# 1 DEA and FDEA models

## 1.1 DEA model

The model of Charnes et al. [1] called CCR model, and the BCC model named after Banker, Charnes and cooper [4] are the frequently used models. The primary difference between the two models is the treatment of returns to scale. The CCR model assumes constant return to scale. The BCC model is more flexible and allows variable returns to scale. Other DEA models exist and all are extensions of the CCR model (see e.g., [13, 14]). In our paper, the focus will be in the CCR model and the technique developed can be adapted for all DEA models. Consider n DMUs, each consumes varying amounts of m different inputs to produce s different outputs. In the models formulation, we denote by

$\mathrm{DMU}_0:$ the target DMU,

$x_{ij}$ : the amount of input i consumed by $\text{DMU}_j$,
$y_{ri}$ : the amount of output r consumed by $\text{DMU}_j$,
$u_i$ : the weight associate to the ith input,
$v_r$ : the weight associate to the rth output.

The programming statements for the (input oriented) CCR model and its dual are respectively :

$$\begin{aligned} Max \quad & \sum_{r=1}^{s} v_r y_{r0} \\ S/T \quad & \sum_{i=1}^{m} u_i x_{i0} = 1,\ i = 1, ....m, \\ & \sum_{i=1}^{m} u_i x_{ij} \geq \textstyle\sum_{r=1}^{s} v_r y_{rj},\ j = 1, .....n, \\ & u_i, v_r \geq 0. \end{aligned} \tag{1}$$

$$\begin{aligned} Min \quad & \theta_0 \\ S/T \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_0 x_{i0},\ i = 1, ....m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{r0}, r = 1, ....s, \\ & \lambda_j \geq 0,\ j = 1, ......n \end{aligned} \tag{2}$$

target DMU ($\text{DMU}_0$) is technically efficient if and only if $\theta^* = 1$. $\theta^*$ is the optimal objective value of model (2), it can be obtained by model (1). From the the definition of Pareto-Koopmans Efficiency [5, 7], at the optimal solution, the target DMU ($\text{DMU}_0$) is fully efficient if and only if $\theta^* = 1$ and it's not possible to make improvement (lower input or higher output) without worsening any other input or output.

## 1.2 FDEA model

In the case of fuzzy data (inputs and outputs), fuzzy set theory can be a powerful alternative to treat the imprecision and the vagueness in DEA models. The result FDEA models take the form of fuzzy linear programming problems. The FCCR and its dual are represented as follow (see [19, 22]):

$$\begin{aligned} Max \quad & \sum_{r=1}^{s} v_r \widetilde{y}_{r0} \\ S/T \quad & \sum_{i=1}^{m} u_i \widetilde{x}_{i0} = 1, i = 1, ....m, \\ & \sum_{i=1}^{m} u_i \widetilde{x}_{ij} \geq \sum_{r=1}^{s} v_r \widetilde{y}_{rj},\ j = 1, .....n, \\ & u_i, v_r \geq 0. \end{aligned} \tag{3}$$

$$\begin{array}{ll} Min & \theta_0 \\ S/T & \sum_{j=1}^{n} \lambda_j \widetilde{x}_{ij} \leq \theta_0 \widetilde{x}_{i0},\ i = 1, ....m, \\ & \sum_{j=1}^{n} \lambda_j \widetilde{y}_{rj} \geq \widetilde{y}_{r0}, r = 1, ....s, \\ & \lambda_j \geq 0,\ j = 1, ......n \end{array} \quad (4)$$

$\widetilde{x}_{ij}$ is the ith fuzzy input utilised by $\mathrm{DMU}_j$, $\widetilde{y}_{rj}$ is the rth fuzzy input produced by $\mathrm{DMU}_j$. Similar to the crisp CCR model, the constraints $\sum_{i=1}^{m} u_i \widetilde{x}_{i0} = 1$ and $\sum_{i=1}^{m} u_i \widetilde{x}_{ij} \geq \sum_{r=1}^{s} v_r \widetilde{y}_{ij}$ in model (3) are used for normalisation of the value $\sum_{r=1}^{s} v_r \widetilde{y}_{r0}$. However, the objective value $\sum_{r=1}^{s} v_r \widetilde{y}_{r0}$ can exceed one since the second and third constraints of (3) are satisfied "possibilistically". That is, since their parameters are fuzzy sets, $\sum_{r=1}^{s} v_r \widetilde{y}_{r0}$ is "approximately equal to one", which implies that $\sum_{i=1}^{m} u_i \widetilde{x}_{ij} / \sum_{r=1}^{s} v_r \widetilde{y}_{ij}$ is "approximately less than or equal to one". The interpretation of constraints of FCCR model is similar to the crisp CCR model. The difference between the two models resides on the manner of resolution. The crisp CCR model can be simply solved by a standard LP solver. For the FCCR model, the resolution is more difficult and requires some ranking methods for ranking fuzzy sets. In what follow, we give the literature review on the approaches proposed to solve FDEA models.The tolerance approach can be found in Sengupta [5], Kahramman and Tolga [6]. The main idea is that uncertainty is incorporated into the DEA models by specifying tolerance levels on constraint violations. The defuzzification approach developed in Lertworasirikul [7] consists, in a first stage, to defuzzify the fuzzy inputs and the fuzzy outputs into crisp values. In a second stage, the crisp values are used in the conventional DEA model. The $\alpha$ based approach can be found in Maeda et al. [14], Kao and Liu [15]. It consists to solve the FDEA model by the use of the parametric programming and the technical of $\alpha$ cut. At a given $\alpha$ level, an interval efficiency is calculated for the target DMU. A number of such intervals can be used to construct the corresponding fuzzy efficiency. Guo and Tanaka [9] developed the fuzzy ranking approach. In the FCCR model, both fuzzy inequality and fuzzy equality are defined by ranking methods so that the resulting model is a bi-level linear programming model. For a given $\alpha$ level, they define a nondominated set to evaluate the efficiency of $\mathrm{DMU}_s$. Lertworasirikul et al. in [19] proposed a possibility approach in which constraints are treated as fuzzy events. The approach transforms FDEA models into possibility DEA models by using possibility measures of fuzzy constraints. In the case of trapezoidal fuzzy data (inputs and outputs), possibility DEA models take the form of linear programming problems. Kao and Liu in [21] developed a method to rank the fuzzy efficiency scores without knowing the exact form of the membership functions. The main idea is to apply the maximizing set–minimizing set technique, which is normally applied when membership functions are known. Via a skilful modelling technique, the requirement of the membership functions is avoided. The efficiency rankings are consequently determined by solving a pair of non linear programs for each DMU. All presented approaches are powerful, but some shortcomings can appear in the way of treatment of fuzzy data in DEA models. For example, with the defuzzification approach the fuzziness in inputs and outputs is effectively ignored. The tolerance approach treats fuzzy inequality and equality instead of fuzzy inputs and fuzzy outputs. The ranking approach of Guo and Tanaka uses only one number at a given level to compare fuzzy efficiencies. With the possibility approach, the numerical computation is more complicated in the case of fuzzy data with non linear membership functions. The next section is devoted to data modelling with LR-fuzzy numbers.

# 2 Data modelling with LR-fuzzy numbers

Several definitions of LR-fuzzy numbers have been published. All of them are variations on the original definition by Dubois and Parade [18 ]. In this section, we will use the definitions and notation in [16]. An LR-fuzzy number is represented as $\widetilde{A}_i = (a_{li}, a_{ui}, c_i, d_i)_{LR}$, where the subset $[a_{li}, a_{ui}]$ consists on the real numbers with the highest chance of realisation, $c_i$ is the left spread, $d_i$ is the right spread and $L$ & $R$ are functions defining the left and the right shapes of the fuzzy number respectively. Its membership function can be represented as:

$$\mu_{\widetilde{A}}(r) = \begin{cases} L(\frac{a_{li}-r}{c_i}), r \leq a_{li} \\ 1,\ a_{li} \leq r \leq a_{ui} \\ R(\frac{r-a_{ui}}{d_i}),\ r \geq a_{ui} \end{cases}$$

A function, $L$ or $R$ , is said to be a reference function of a fuzzy number $\widetilde{A} = (x, \mu_{\widetilde{A}}(x))$ , if and only if the following conditions are satisfied:

$L, R : [0, +\infty[ \to [0, 1]$,

$L(x) = L(-x)$, $R(x) = R(-x)$,

$L(0) = 1$, $R(0) = 1$, and

$L(x)$ and $R(x)$ are strictly decreasing and upper semi-continuous on supp $(\widetilde{A})$ .

Some particular cases are triangular and trapezoidal fuzzy numbers, for which, are linear function. In [18], non linear reference function could be classified as follow:

Power : $Sp(x) = \max\{0, 1 - x_p\}, p > 0,$

Exponential :$Sp(x) = \max\{0, a_p(1 - \exp[-p(1-x)])\},$
$ap = [1 - \exp(-p)]^{-1}$

Rational :$Sp(x) = 1/(1 + x^p), p \geq 1.$

More details on reference function can be found in [16]. In what follow, we give some results for LR-fuzzy numbers that belong to the same family and have different shapes.

**Definition 1** (Dubois and Parade [18]). Let $\widetilde{M}$ and $\widetilde{N}$ be two fuzzy numbers. Then, $\widetilde{M} \succ \widetilde{N} \Longleftrightarrow \widetilde{M} \cup \widetilde{N} = \widetilde{M}$, where $\widetilde{M} \cup \widetilde{N}$ represents the maximum of those fuzzy numbers.

**Prposition 1** (Ramik and Rimanek [17]: Let $\widetilde{M}$ and $\widetilde{N}$ be two fuzzy numbers. Then $\widetilde{M} \cup \widetilde{N}$ if and only if $\inf\{s : \mu_{\widetilde{M}}(s) \geq w\} \geq \inf\{t : \mu_{\widetilde{M}}(t) \geq w\}$ and $\sup\{s : \mu_{\widetilde{M}}(s) \geq w\} \geq \sup\{t : \mu_{\widetilde{M}}(t) \geq w\}$ hold for all grades of membership $w$ in $[0, 1]$.

**Definition 2** (Tanaka et al. [9]). Let $\widetilde{M}$ and $\widetilde{N}$ be two fuzzy numbers and $h$ a real number, $h \in [0, 1]$, then $\widetilde{M} \succcurlyeq^h \widetilde{N}$ if and only if $\inf\{s : \mu_{\widetilde{M}}(s) \geq w\} \geq \inf\{t : \mu_{\widetilde{M}}(t) \geq w\}$ and $\sup\{s : \mu_{\widetilde{M}}(s) \geq w\} \geq$
hold.

Hence, for two LR-fuzzy numbers $\widetilde{M} = (m_l, m_u, c_1, d_1)$ and $\widetilde{N} = (n_l, n_u, c_2, d_2)$ at a given possibility level $h$: $\widetilde{M} \succcurlyeq^h \widetilde{N}$ , then its necessary to check $m_1 - c_1 L^{-1}(w) \geq n_1 - c_2 \acute{L}^{-1}(w)$ and $m_u + d_1 R^{-1}(w) \geq n_u + d_2 \acute{R}^{-1}(w)$, $w \in [h, 1]$

# 3 Solving FDEA models

We Consider that the fuzzy inputs and outputs in the FCCR model represented by (4) in the second section can be expressed as LR-fuzzy numbers, and have different power reference functions. To be more clearly we give the following notation:

$\widetilde{x}_{ij} = (x_{lij}, x_{uij}, a_{ij}, b_{ij})$: The input i consumed by DMUj,

$\widetilde{y}_{rj} = (y_{lrj}, y_{urj}, c_{rj}, d_{rj})$: The output r consumed by DMUj,

$L_{ij}^{-1}(h)$: The inverse of the left reference function of $\widetilde{x}_{ij}$ .

$R_{ij}^{-1}(h)$: The inverse of the right reference function of $\widetilde{x}_{ij}$ .

$\acute{L}_{rj}^{-1}(h)$: The inverse of the left reference function of $\widetilde{y}_{rj}$ .

$\acute{R}_{rj}^{-1}(h)$: The inverse of the right reference function of $\widetilde{y}_{rj}$ .

## 3.1 Solving the dual of FCCR

Since fuzzy inputs and fuzzy outputs are LR fuzzy numbers, the constraints in (4) can be considered as inequalities between LR fuzzy numbers, and the use of the ordering relation in [9] allows us to convert each fuzzy constraint into two crisp inequalities. Thus, model (4) can be transformed in the following crisp linear programming problem:

$$
\begin{aligned}
&Min \;\; \theta_0 \\
&\sum_{j=1}^{n}\left(x_{lij}-a_{ij}L_{ij}^{-1}(h)\right)\lambda_j \le \left(x_{li0}-a_{i0}L_{i0}^{-1}(h)\right)\theta_0,\; i=1,.....,m,\\
&\sum_{j=1}^{n}\left(x_{uij}+b_{ij}R_{ij}^{-1}(h)\right)\lambda_j \le \left(x_{ui0}+b_{i0}R_{i0}^{-1}(h)\right)\theta_0,\; i=1,.....,m, \qquad \mathbf{(5)}\\
&\sum_{j=1}^{n}\left(y_{lrj}-c_{rj}\acute{L}_{rj}^{-1}(h)\right)\lambda_j \ge \left(y_{lr0}-c_{rj}\acute{L}_{r0}^{-1}(h)\right),\; r=1,.....,s,\\
&\sum_{j=1}^{n}\left(y_{urj}+d_{rj}\acute{R}_{rj}^{-1}(h)\right)\lambda_j \ge \left(y_{ur0}+d_{r0}\acute{R}_{r0}^{-1}(h)\right),\; r=1,.....,s,\\
&\lambda_j \ge 0,\;\; h\in[0,1],\;\; j=1,.....,n.
\end{aligned}
$$

We note that if the fuzzy inputs and fuzzy outputs are triangular or trapezoidal, then, $L_{ij}^{-1}(h)=R_{ij}^{-1}(h)=\acute{L}_{rj}^{-1}(h)=\acute{R}_{rj}^{-1}(h)$.
optimal value of (5) provides an evaluation of the efficiency of the target DMU ($\mathrm{DMU}_0$) in witch all the possible values of different variables are considered and the decision maker can obtain efficiency scores with respect to a given possibility level. The value efficiency at a given possibility level h is crisp.

**Prposition 2** The efficiency score is a nonincreasing function of possibility level $h$ .

**Proof** let $S=(\lambda_1^*,....,\lambda_n^*,\theta^*)$ is the optimal solution of (4) at the possibility level $h$, then, we have $\sum_{j=1}^{n}\lambda_j\widetilde{x}_{ij}\le\theta_0\widetilde{x}_{i0}$, $i=1,....m$, and $\sum_{j=1}^{n}\lambda_j\widetilde{y}_{rj}\ge\widetilde{y}_{r0}$, $r=1,....s$, for all possibility level $\acute{h}\in[h,1]$ . Therefore, $S$ is a feasible solution of (3) for all $\acute{h}$ such that $h\le\acute{h}\le 1$ . Consequently, the optimal value of model (4), which is a minimisation problem, at possibility level $\acute{h}$ will be greater than or equal to $\theta_0^*$

**prposition 3** $\mathrm{DMU}_0$ is called fuzzy efficient if and only if it is efficient at least at one possibility level $h$. Otherwise, it is fuzzy inefficient.

Here, we define two fuzzy sets $\widetilde{E}$ and $\widetilde{\bar{E}}$ for the fuzzy efficient and fuzzy inefficient DMUs respectively.

For a given DMUj, the membership functions are givenby

$$
\mu_{\widetilde{E}}(DMU_j)=\begin{cases} *0 \text{ if } \mathrm{DMU_j} \text{ is inefficient at all possibility levels } h.\\ *\sup\left\{h:\theta_j^*(h)=1\right\} \text{ if } \mathrm{DMU_j} \text{ is efficient at some}\\ \quad\text{possibility levels } h.\end{cases}
$$

$$
\mu_{E^{\simeq}}(DMU_j)=\begin{cases} *1-\sup\left\{h:\theta_j^*(h)=1\right\} \text{ if } \mathrm{DMU_j} \text{ is efficient .}\\ \text{at some possibility levels } h.\\ *\;1 \text{ if } \mathrm{DMU_j} \text{ is inefficient at all possibility levels } h\;.\end{cases}
$$

**Definition 3** $\mathrm{DMU}_0$ is called fuzzy nondominated efficient if and only if it is efficient at all possibility levels $h$ .

**Example 1** Suppose that there are 5 DMU$s$, two fuzzy inputs and two fuzzy outputs with different power reference functions. Values of data and reference functions are shown in table 1 and 2, respectively:

Table 1 value of fuzzy inputs and fuzzy output

| $\text{DMU}_j$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Input 1 | (1;0.25;025) | (2;0.2;0.2) | (3;0.5;1) | (6;2;1) | (1;0.25;0.25) |
| Input 2 | (3;0.5;1) | (2;0.2;0.2) | (2;0.2;0.2) | (4;1.5;1) | (3;0.5;1) |
| Output 1 | (2;0.2;0.2) | (1;0.25;0.25) | (1;0.25;0.25) | (2;0.2;0.2) | (1;0.25;0.25) |
| Output 2 | (6;2;1) | (3.5;0.5;1) | (2;0.2;0.2) | (6;2;1) | (3.5;0.5;1) |

Table 2 reference functions of fuzzy input and fuzzy output

| Inputs | $L(K)$ | $R(K)$ | outputs | $L'(K)$ | $R'(K)$ |
|---|---|---|---|---|---|
| $\widetilde{x}_{11}$ | $1-k^8$ | $1-k^3$ | $\widetilde{y}_{11}$ | $1-k^3$ | $1-k^3$ |
| $\widetilde{x}_{12}$ | $1-k$ | $1-k^2$ | $\widetilde{y}_{12}$ | $1-k^4$ | $1-k$ |
| $\widetilde{x}_{21}$ | $1-k^3$ | $1-k^3$ | $\widetilde{y}_{21}$ | $1-k^8$ | $1-k^3$ |
| $\widetilde{x}_{22}$ | $1-k^3$ | $1-k^3$ | $\widetilde{y}_{22}$ | $1-k^3$ | $1-k^3$ |
| $\widetilde{x}_{31}$ | $1-k$ | $1-k^2$ | $\widetilde{y}_{31}$ | $1-k^8$ | $1-k^3$ |
| $\widetilde{x}_{32}$ | $1-k^3$ | $1-k^3$ | $\widetilde{y}_{32}$ | $1-k^5$ | $1-k^3$ |
| $\widetilde{x}_{41}$ | $1-k^4$ | $1-k$ | $\widetilde{y}_{41}$ | $1-k^3$ | $1-k^3$ |
| $\widetilde{x}_{42}$ | $1-k$ | $1-k^2$ | $\widetilde{y}_{42}$ | $1-k^4$ | $1-k$ |
| $\widetilde{x}_{51}$ | $1-k^8$ | $1-k^3$ | $\widetilde{y}_{51}$ | $1-k^8$ | $1-k^3$ |
| $\widetilde{x}_{52}$ | $1-k$ | $1-k^2$ | $\widetilde{y}_{52}$ | $1-k$ | $1-k^2$ |

The application of model (5) provides the efficiency values registered in table 3. The efficiency value of each DMU is a nonincreasing function of possibility level $h$ . $\text{DMU}_1$ is efficient at all possibility levels, and then it is called a fuzzy nondominated DMU. $\text{DMU}_2$, $\text{DMU}_3$ and $\text{DMU}_4$ are efficient at possibility levels (0, 0.25, 0.5, 75), (0, 0.25) and (0) respectively, then, they are fuzzy efficient. $\text{DMU}_5$ is inefficient at all possibility levels, then, it is called fuzzy inefficient.

Table 3 efficiency values for example 1

| $\alpha$ | $DMU_1$ | $DMU_2$ | $DMU_3$ | $DMU_4$ | $DMU_5$ |
|---|---|---|---|---|---|
| 0.0 | 1.000 | 1.000 | 1.000 | 1.000 | 0.785 |
| 0.25 | 1.000 | 1.000 | 1.000 | 0.913 | 0.772 |
| 0.50 | 1.000 | 1.000 | 0.975 | 0.846 | 0.768 |
| 0.75 | 1.000 | 1.000 | 0.930 | 0.806 | 0.761 |
| 1.00 | 1.000 | 0.875 | 0.750 | 0.750 | 0.583 |

## 3.2 Solving the primal of FCCR

The primal of FCCR in (3) takes the form of a linear programming problem with fuzzy coefficients in the objective function and in the constraints. We use the ordering relation between fuzzy numbers to solve this problem. First, let us consider the objective function $\max \sum_{r=1}^{s} v_r \widetilde{y}_{r0}$. Because of the efficiency value is positive; the objective function can be regarded as a constraint $(\max \sum_{r=1}^{s} v_r \widetilde{y}_{r0} \geq 0\ )$. Using the ordering relation in definition 2, this constraint is decomposed into two crisp relations as follow; $\max \sum_{r=1}^{s} v_r \left( y_{lr0} - c_{r0} \acute{L}_{r0}^{-1}(h) \right)$ and $\max \sum_{r=1}^{s} v_r \left( y_{ur0} + d_{r0} \acute{R}_{r0}^{-1}(h) \right), h \in [0,1]$. Then, $\max \sum_{r=1}^{s} v_r \widetilde{y}_{r0}$ is equivalent to maximise $\sum_{r=1}^{s} v_r \left( y_{lr0} - c_{r0} \acute{L}_{r0}^{-1}(h) \right)$ and $\sum_{r=1}^{s} v_r \left( y_{lr0} - c_{r0} \acute{L}_{r0}^{-1}(h) \right)$ simultaneously. A weighted function

$\lambda_1 \sum_{r=1}^{s} v_r \left( y_{lr0} - c_{r0} \acute{L}_{r0}^{-1}(h) \right) + \lambda_2 \sum_{r=1}^{s} v_r \left( y_{ur0} + d_{r0} \acute{R}_{r0}^{-1}(h) \right)$ with $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$ is used to obtain some compromise solution. Here, the values of $\lambda_1$ and $\lambda_2$ reflect the opinion of the decision maker, we consider three cases; optimistic if $\lambda_2 = 1$, pessimistic if $\lambda_1 = 1$ and indifferent if $\lambda_1 = \lambda_2$ . Then the following objective function is obtained:

$$\max \lambda_1 \sum_{r=1}^{s} v_r \left( y_{lr0} - c_{r0} \acute{L}_{r0}^{-1}(h) \right) + \lambda_2 \sum_{r=1}^{s} v_r \left( y_{ur0} + d_{r0} \acute{R}_{r0}^{-1}(h) \right) \qquad \mathbf{(6)}$$

Next, let us consider the normalisation equality $\sum_{i=1}^{m} u_i \widetilde{x}_{i0} = 1$ . We transform this relation into two crisp equalities as follow:

$$\sum_{i=1}^{m} u_i \left( x_{li0} - a_{i0} L_{i0}^{-1}(h) \right) = 1, \qquad \mathbf{(7)}$$
$$\sum_{i=1}^{m} u_i \left( x_{ui0} + b_{io} R_{i0}^{-1}(h) \right) = 1$$

weights $u_i$ cannot be found out to satisfy equalities (7) simultaneously. To avoid this difficulty, we consider each equality as a goal to achieve. Thus, a goal programming problem must be constructed:

$$Min\ d_1^+ + d_1^- + d_2^+ + d_2^-$$

$$\sum_{i=1}^{m} u_i \left( x_{li0} - a_{i0} L_{i0}^{-1}(h) \right) - d_1^+ + d_1^- = 1,\ i = 1, \ldots\ldots, m,$$
$$\sum_{i=1}^{m} u_i \left( x_{ui0} + b_{i0} R_{i0}^{-1}(h) \right) - d_2^+ + d_2^- = 1,\ i = 1, \ldots\ldots, m, \quad \textbf{(8)}$$
$$d_1^+ \geq 0,\ d_2^+ \geq 0,\ d_1^- \geq 0, d_2^- \geq 0$$
$$u_i \geq 0, h \in [0,1],$$

$d_i^+$ et $d_i^-$ are respectively the negative and positive deviation corresponding to goal $g_j$, $j = 1, 2$ . Solving problem (8) provides the values of $\rho_i \left( \rho_i = d_i^+ - d_i^- \right)$. These values are added to the left sides of equalities (7) and finally the equality $\sum_{i=1}^{m} u_i \widetilde{x}_{i0} = 1$ is converted as follow:

$$\sum_{i=1}^{m} u_i \left( x_{li0} - a_{i0} L_{i0}^{-1}(h) \right) - \rho_1 = 1, \quad \textbf{(9)}$$
$$\sum_{i=1}^{m} u_i \left( x_{ui0} + b_{i0} R_{i0}^{-1}(h) \right) - \rho_2 = 1,$$

The use of ordering relation in definition 2, at a given possibility level $h$ , allows us to transform the constraint $\sum_{i=1}^{m} u_i \widetilde{x}_{ij} \geq \sum_{r=1}^{s} v_r \widetilde{y}_{ij}$ into two the following crisp relations; $\sum_{i=1}^{m} u_i \left( x_{lij} - a_{ij} L_{ij}^{-1}(h) \right) \geq \sum_{r=1}^{s} v_r \left( y_{lrj} - c_{rj} \acute{L}_{rj}^{-1}(h) \right)$ and $\sum_{i=1}^{m} u_i \left( x_{uij} + b_{ij} R_{ij}^{-1}(h) \right) \geq \sum_{r=1}^{s} v_r \left( y_{urj} + d_{rj} \acute{R}_{rj}^{-1}(h) \right)$.Finally, the primal of fuzzy CCR can be transformed into the following crisp linear problem:

$$\max \lambda_1 \sum_{r=1}^{s} v_r \left( y_{lr0} - c_{r0} \acute{L}_{r0}^{-1}(h) \right) + \lambda_2 \sum_{r=1}^{s} v_r \left( y_{ur0} + d_{r0} \acute{R}_{r0}^{-1}(h) \right)$$
$$\sum_{i=1}^{m} u_i \left( x_{li0} - a_{i0} L_{i0}^{-1}(h) \right) - \rho_1 = 1$$
$$\sum_{i=1}^{m} u_i \left( x_{ui0} + b_{i0} R_{i0}^{-1}(h) \right) - \rho_2 = 1$$
$$\sum_{i=1}^{m} u_i \left( x_{lij} - a_{ij} L_{ij}^{-1}(h) \right) \geq \sum_{r=1}^{s} v_r \left( y_{lrj} - c_{rj} \acute{L}_{rj}^{-1}(h) \right) \quad \textbf{(10)}$$
$$\sum_{i=1}^{m} u_i \left( x_{uij} + b_{ij} R_{ij}^{-1}(h) \right) \geq \sum_{r=1}^{s} v_r \left( y_{urj} + d_{rj} \acute{R}_{rj}^{-1}(h) \right)$$
$$u_i \geq 0,\ v_r \geq 0,\ h \in [0,1]$$

The optimal value of (10) provides a crisp value of the efficiency of the target DMU ($DMU_0$) in witch all the possible values of different variables are considered. The decision maker can obtain efficiency scores with respect to a given possibility level. We note here that our methodology to solve the primal of FCCR differ from the method of Guo and Tanaka in [9] in the three following points. Firstly, Guo and Tanaka used a triangular and symmetric fuzzy data. In our model, we can use different forms of fuzzy data. Secondly, they handled the constraint $\sum_{i=1}^{m} u_i\widetilde{x}_{i0} = 1$ by the resolution of a linear programming problem. In our approach, we converted this equality into two crisp constraints by the use of a goal programming problem. Thirdly, they obtained fuzzy efficiency scores. Our model gives crisp values. In what follow, we will use the example of Guo and Tanaka to illustrate our approach.

**Example 2** The data of this example are listed in table 4. There are two fuzzy inputs and two fuzzy outputs. These fuzzy inputs and outputs have symmetrical triangular membership functions. They are denoted by (a, c, d) where a is the centre, c is the left spread and d is the right spread.

Table 4 value of fuzzy inputs and fuzzy output

| $DMU_j$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Input 1 | (4.0;0.5) | (2.9;0.0) | (4.9;0.5) | (4.1;0.7) | (6.5;0.6) |
| Input 2 | (2.1;0.2) | (1.5;0.1) | (2.6;0.4) | (2.3;0.1) | (4.1;0.5) |
| Output 1 | (2.6;0.2) | (2.2;0.0) | (3.2;0.5) | (2.9;0.4) | (5.1;0.7) |
| Output 2 | (4.1;0.3) | (3.5;0.2) | (5.1;0.8) | (5.7;0.2) | (7.4;0.9) |

Table 5 shows that, the efficiency score of a DMU is an increasing function of possibility levels $h$ except $DMU_5$ witch is efficient at all possibility levels. $DMU_3$ and $DMU_1$ have efficiency scores less than one at all possibility levels, then they are called fuzzy inefficient. $DMU_2$ and $DMU_4$ are efficient at possibility level one. This means that they are fuzzy efficient.

It was shown in [9] that when using Guo and Tanaka's fuzzy ranking approach, $DMU_2$ and $DMU_4$ fell in all $\alpha$-possibilistic nondominated sets. In addition, using the possibility approach [19], $DMU_2$, $DMU_4$ and $DMU_5$ are efficient for five different possibility levels

Table 5 efficiency values for example 2

| $\alpha$ | $DMU_1$ | $DMU_2$ | $DMU_3$ | $DMU_4$ | $DMU_5$ |
|---|---|---|---|---|---|
| 0.0 | 0.747 | 0.950 | 0.742 | 0.962 | 1.070 |
| 0.25 | 0.769 | 0.960 | 0.768 | 0.968 | 1.053 |
| 0.50 | 0.790 | 0.971 | 0.796 | 0.987 | 1.033 |
| 0.75 | 0.814 | 0.983 | 0.860 | 0.989 | 1.015 |
| 1.00 | 0.855 | 1.000 | 0.860 | 1.000 | 1.000 |

# Conclusion

In this paper, an approach based on the ordering relations between fuzzy numbers for solving fuzzy DEA models has been developed to provide an alternative treatment of fuzzy DEA models. In this approach, fuzzy inputs and fuzzy outputs are represented by LR-fuzzy numbers with different power reference functions. For the dual of FCCR, the two fuzzy constraints were transformed in four crisp constraints. Then the resulting model takes the form of crisp linear programming problem. A numerical example in witch fuzzy inputs and fuzzy outputs has different power reference functions are used to demonstrate the implementation of the proposed approach.

To solve the primal of FCCR, the objective function were considered as fuzzy constraint. This constraint was discovered into crisp function with the use of a weighting function. The fuzzy normalization equality was transformed into two crisp equalities by the use of a goal programming problem. Then the resulting model takes the form of crisp linear programming problem. A numerical example with symmetric triangular reference function was used.

Another interesting topic for future is the resolution of FDEA models in witch fuzzy inputs and fuzzy outputs have exponential or rational reference functions.

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