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Recursive Relations for the Number of Spanning Trees

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Abstract

In this paper, we find recursive relations $t(\mathcal{L}_n) = 4t(\mathcal{L}_{n-1}) - t(\mathcal{L}_{n-2})$, $t(\mathcal{F}_n) = 3t(\mathcal{F}_{n-1}) - t(\mathcal{F}_{n-2})$, and $t(W_n) = t(W_{n-1}) + t(\mathcal{F}_n) + t(\mathcal{F}_{n-1})$, for determining the number of spanning trees in ladders $\mathcal{L}_n$, fans $\mathcal{F}_n$, and wheels W_n.

Mathematics Subject Classification: Primary 05C05, Secondary 05C30

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1 Introduction

All graphs in this paper are finite, undirected, and simple (i.e. without loops or multiple edges). For a graph G, a *spanning tree* in G is a tree which has the same vertex set as G. The number of spanning trees in a graph (network) G, denoted by $t(G)$, is an important invariant of the graph (network). It is also an important measure of reliability of a network.

A famous and classic result on the study of $t(G)$ is the following theorem, known as the *Matrix tree Theorem* [3]. The *Laplacian matrix* (also called *Kirchhoff matrix*) of a graph G is defined as $\mathcal{L}(G) = \mathcal{D}(G) - \mathcal{A}(G)$, where $\mathcal{D}(G)$ and $\mathcal{A}(G)$ are the degree matrix and the adjacency matrix of G, respectively.

Theorem 1.1. *(Matrix tree Theorem) For every connected graph G, $t(G)$ is equal to any cofactor of $\mathcal{L}(G)$.* □

However this theorem is not feasible for large graphs, and various techniques are extended to find the number of spanning trees in different classes of graphs.

Let us first review some methods of combining graphs.

The *union* of graphs G and H is the graph $G \bigcup H$ with vertex set $V(G) \bigcup V(H)$ and edge set $H(G) \bigcup E(H)$. If G and H are disjoint, we refer to their union as a *disjoint union*, denoted by $G + H$. The *join* of two graphs G and H, $G \bigvee H$, is obtained from the disjoint union of G and H by additionally joining every vertex of G to every vertex of H.

The join $W_n = C_n \bigvee K_1$ of a cycle C_n and a single vertex is referred to as a *wheel* with n *spokes*. Similarly, the join $\mathcal{F}_n = P_n \bigvee K_1$ of a path P_n and a single vertex is called a *fan*.

The *cartesian product* of graphs G and H is the graph $G \,\square\, H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2$. The notation used for the cartesian product reflects this fact. The cartesian product $\mathcal{L}_n = P_2 \,\square\, P_n$ is called a *ladder*.

In the next section, we derive recursive relations $t(\mathcal{L}_n) = 4t(\mathcal{L}_{n-1}) - t(\mathcal{L}_{n-2})$, $t(\mathcal{F}_n) = 3t(\mathcal{F}_{n-1}) - t(\mathcal{F}_{n-2})$, and $t(W_n) = t(W_{n-1}) + t(\mathcal{F}_n) + t(\mathcal{F}_{n-1})$, for evaluating the number of spanning trees in ladders, fans, and wheels.

2 Main Results

There is a simple and elegant recursive formula for the number of spanning trees in a graph. It involves the operation of contraction of an edge, which we now introduce. An edge e of a graph G is said to be *contracted* if it is deleted and its ends are identified. The resulting graph is denoted by $G.e$. Also we denote by $G - e$ the graph obtained from G by deleting the edge e.

Theorem 2.1. *[3] Let G be a graph (multiple edges are allowed in here). Then for any edge e*

$$t(G) = t(G - e) + t(G.e).$$

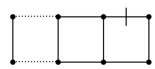
□

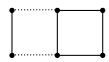
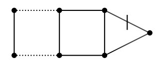
Now by applying this theorem, we obtain the expressed recursive relations.

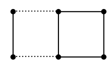
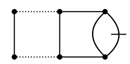
Theorem 2.2. *The number of spanning trees of the ladder $\mathcal{L}_n$ ($n \geq 3$) satisfies the following recursive relation:*

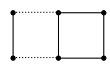
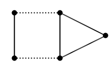
$$t(\mathcal{L}_n) = 4t(\mathcal{L}_{n-1}) - t(\mathcal{L}_{n-2}).$$

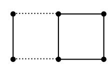
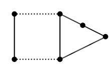
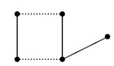
Proof. By using theorem above we get

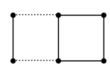
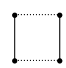
$t(\mathcal{L}_n) = t($  $)$

$= t($  $) + t($  $)$

$= 2t($  $) + t($  $)$

$= 3t($  $) + t($  $)$

$= 3t($  $) + t($  $) - t($  $)$

$= 4t($  $) - t($  $)$

$= 4t(\mathcal{L}_{n-1}) - t(\mathcal{L}_{n-2}).$

On the other hand, we know that $t(\mathcal{L}_1) = 1$, and $t(\mathcal{L}_2) = 4$. Consequently by solving this recursion we obtain:

Corollary 2.3. *The number of spanning trees in the ladder $\mathcal{L}_n$ $(n \geq 1)$ is equal to:*

$$t(\mathcal{L}_n) = \frac{\sqrt{3}}{6}((2+\sqrt{3})^n - (2-\sqrt{3})^n).$$

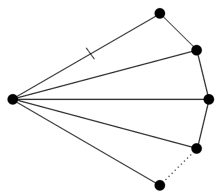
□

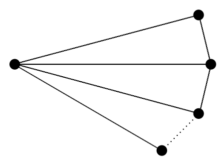
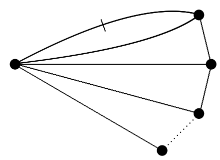
This corollary is due to Sedlacek ([6]), but it is derived from a different approach. Similarly, we obtain a recursive relation for calculating the number of spanning trees in a fan.

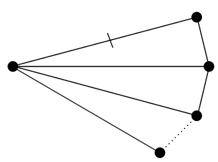
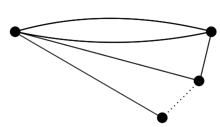
Theorem 2.4. *For the fan $\mathcal{F}_n$ $(n \geq 3)$*

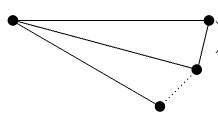
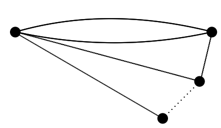
$$t(\mathcal{F}_n) = 3t(\mathcal{F}_{n-1}) - t(\mathcal{F}_{n-2}).$$

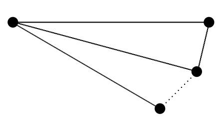
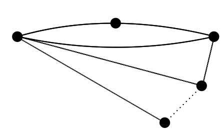
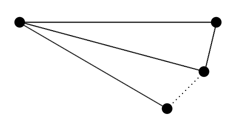
Proof. Applying theorem 2.1 gives

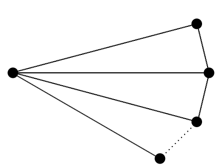
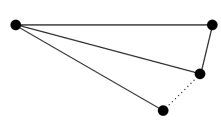
$t(\mathcal{F}_n) = t($  $)$

$= t($  $) + t($  $)$

$= 2t($  $) + t($  $)$

$= 2t($  $) + 3t($  $)$

$= 2t($  $) + 3\Big(t($  $) - t($  $)\Big)$

$= 3t($  $) - t($  $)$

$= 3t(\mathcal{F}_{n-1}) - t(\mathcal{F}_{n-2})$

Since $t(\mathcal{F}_1) = 1$, and $t(\mathcal{F}_2) = 3$, the following corollary follows, which is also proved in [1] by applying the matrix tree theorem.

Corollary 2.5.

$$t(\mathcal{F}_n) = F_{2n} = \frac{1}{\sqrt{5}}((\frac{3+\sqrt{5}}{2})^n - (\frac{3-\sqrt{5}}{2})^n),\ n \geq 1,$$

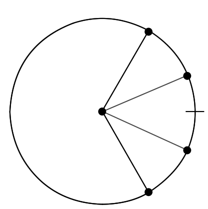
where F_n denotes the nth Fibonacci number. That is, $F_{n+2} = F_{n+1} + F_n$, for $n \geq 1$ with $F_1 = F_2 = 1$. □

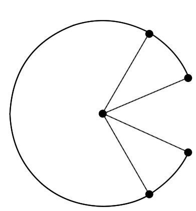
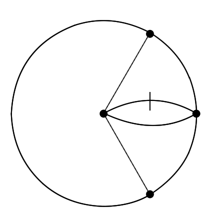
Now we derive a recursion for enumerating the number of spanning trees in a wheel.

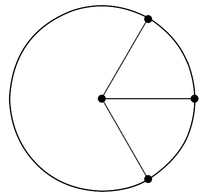
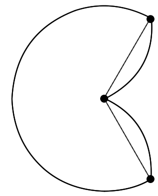
Theorem 2.6. *For the wheel W_n $(n \geq 4)$ we have*

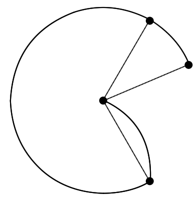
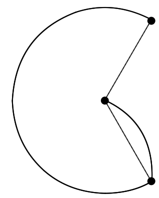
$$t(W_n) = t(W_{n-1}) + t(\mathcal{F}_n) + t(\mathcal{F}_{n-1}).$$

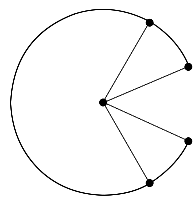
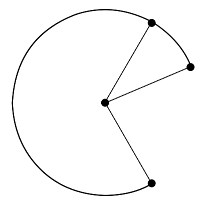
Proof. Again theorem 2.1 implies that

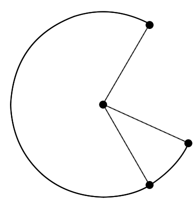
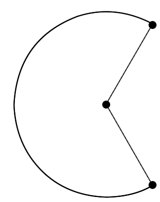
$t(W_n) = t($  $)$

$= t($  $) + t($  $)$

$= t(\mathcal{F}_n) + t($  $) + t($  $)$

$= t(\mathcal{F}_n) + t(W_{n-1}) + t($  $) - t($  $)$

$= t(\mathcal{F}_n) + t(W_{n-1}) + t($  $) - t($  $)$

$- t($  $) + t($  $)$

$$= t(W_{n-1}) + 2t(\mathcal{F}_n) - 2t(\mathcal{F}_{n-1}) + t(\mathcal{F}_{n-2})$$

$$= t(W_{n-1}) + t(\mathcal{F}_n) + t(\mathcal{F}_{n-1}).$$

If we define C_2 as two parallel edges, then this recursive relation holds also for $n = 2$. Thus, since $t(W_2) = 5$, we get:

Corollary 2.7. *The number of spanning trees of the wheel W_n ($n \geq 1$) is equal to:*

$$t(W_n) = L_{2n} - 2 = F_{2n+2} - F_{2n-2} - 2 = (\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n - 2,$$

where L_n denotes the nth Lucas number. That is, $L_{n+2} = L_{n+1} + L_n$, for $n \geq 1$ with $L_1 = 1$ and $L_2 = 3$. □

This formula first proved by Sedlacek ([5]) and later by Myers ([4]), using the matrix tree theorem.

References

[1] Z. R. Bogdanowicz, Formulas for the Number of Spanning Trees in a Fan, *Applied Mathematical Sciences,* Vol. 2, No. 16 (2008), 781-786.

[2] W. Feussner, Über Stromverzweigung in Netzförmigen Leitern, *Annalen der Physik,* 9 (1902), 1304-1329.

[3] G. Kirchhoff, Über die Auflösung der Gleichungen auf, welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird, *Ann. Phy. Chem.*, 72 (1847), 497-508.

[4] B. R. Myers, Number of Spanning Trees in a Wheel, *IEEE Transactions on Circuit Theory,* CT-18 (1971), 280-282.

[5] J. Sedlacek, On the Skeletons of a Graph or Digraph, *Proceedings of the Calgary International Conference of Combinatorial Structures and Their Applications,* Gordon and Breach (1970), 387-391.

[6] J. Sedlacek, On the Spanning Trees of Finite Graphs, *Cas. Pestovani Mat.,* 94 (1969) 217-221.

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