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Application of Homotopy Perturbation Method to Some Nonlinear Problems

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Abstract

In this paper, we obtain the series solutions to three well known equations arising in different fields of science. We apply homotopy perturbation method (HPM) to Burgers equation, the regularized long wave equation and the modified Korteweg-de Vries equation. In each problem, applying HPM, we obtain the Taylor expansion of the exact solution which are convergent in desired domains.

Mathematics Subject Classification: 35C05, 35Q53, 35A20

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1 Introduction

In recent years, the homotopy perturbation method (HPM), first proposed by Ji Huan He [13, 14], has successfully been applied to solve many types of linear and nonlinear functional equations. This method, which is a combination of homotopy in topology and classic perturbation techniques, provides us with a convenient way to obtain analytic or approximate solutions to a wide variety of problems arising in different fields.
Dr. He used HPM to solve Lighthill equation [13], Duffing equation [15] and Blasius equation [16], then the idea go through and has been used to solve nonlinear wave equations [17], boundary value problems [18], integral equations [12, 11, 4], Klein-Gordon and sine-Gordon equations [22], Schrödinger equation [5], Emden-Fowler type equations [8] and many other problems. This wide variety of applications show the power of HPM in solving functional equations.

In this work, we apply HPM to three important problems which has been of great importance among researchers. The Burgers equation, the regularized

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long wave equation and the modified Korteweg-de Vries equation are the ones that we consider.

The Burgers equation serves as a useful model for many interesting problems in applied mathematics. It effectively models certain problems of a fluid flow nature, in which either shocks or viscous dissipation is a significant factor. The first steady-state solution of Burgers equation were given by Bateman [2] in 1915. However, the equation gets its name from the extensive research of Burgers [7] beginning in 1939. It can be used as a model for any nonlinear wave propagation problem subject to dissipation [10]. Depending on the problem being modeled, this dissipation may result from viscosity, heat conduction, mass diffusion, thermal radiation, chemical reaction, or other sources.

The regularized long wave (RLW) equation is a partial differential equation serving as a model for nonlinear dispersive waves which has applications in many areas, e.g. ion-acoustic waves in plasma, magnetohydrodynamics waves in plasma, longitudinal dispersive waves in elastic rods, pressure waves in liquid-gas bubble mixtures and rotating flow down a tube. Mathematical theory for the RLW equation was developed in [3, 6]. Various numerical techniques have been proposed to solve the RLW equation. Some of this methods are finite difference, Fourier, finite element, collocation, spline and variational methods.

The modified Korteweg-de Vries (mKdV) equation plays an important role in many nonlinear science fields [21, 9]. It has been used to describe acoustic waves in certain anharmonic lattices [1], Alfvén waves in a collisionless plasma [19], Schottky barriers transmission lines [25], models of traffic congestion [20], etc.

2 Mathematical formulation

The Burgers equation is presented as

$$\begin{aligned} &y_t + yy_x - y_{xx} = 0, \qquad\qquad x \in \mathbb{R} \\ &y(x,0) = \frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{4}x). \end{aligned} \tag{1}$$

The exact solution of this equation is [23]

$$y(x,t) = \frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{4}(x - \frac{1}{2}t)). \tag{2}$$

Expanding (2) using Taylor expansion about $(x,t)=(0,0)$ gives

$$\begin{aligned} y(x,t) = \; & \frac{1}{2} - \frac{1}{8}x + \frac{1}{16}t - \frac{1}{3072}t^3 + \frac{1}{512}xt^2 - \frac{1}{256}x^2t + \frac{1}{384}x^3 + \frac{1}{491520}t^5 \\ & - \frac{1}{49152}xt^4 + \frac{1}{12288}x^2t^3 - \frac{1}{6144}x^3t^2 + \frac{1}{6144}x^4t - \frac{1}{15360}x^5 \\ & - \frac{17}{1321205760}t^7 + \frac{17}{94371840}xt^6 - \frac{17}{15728640}x^2t^5 + \frac{17}{4718592}x^3t^4 \\ & - \frac{17}{2359296}x^4t^3 + \frac{17}{1966080}x^5t^2 - \frac{17}{2949120}x^6t + \frac{17}{10321920}x^7 + \cdots . \end{aligned} \tag{3}$$

The regularized long wave (RLW) equation is formulated as follows:

$$\begin{aligned} & y_t + y_x + yy_x - y_{xxt} = 0, \quad x \in \mathbb{R} \\ & y(x,0) = 3\alpha \operatorname{sech}(\beta x). \end{aligned} \tag{4}$$

where $\alpha > 0$ is a constant and $\beta = \frac{1}{2}(\alpha/(\alpha+1))^{(1/2)}$.

We consider the modified Korteweg-de Vries (mKdV) equation

$$\begin{aligned} & y_t + 6y^2y_x + y_{xxx} = 0, \quad x \in \mathbb{R} \\ & y(x,0) = 2\operatorname{sech}(2x). \end{aligned} \tag{5}$$

It has the exact solution as follows [24]

$$y(x,t) = 2\operatorname{sech}(2x - 8t). \tag{6}$$

The Taylor expansion of (6) about $(x,t)=(0,0)$ is

$$\begin{aligned} y(x,t) = \; & 2 + \frac{5120}{3}t^4 + \frac{20}{3}x^4 - 4x^2 + 32xt - 64t^2 - \frac{320}{3}x^3t + 640x^2t^2 \\ & - \frac{5120}{3}xt^3 - \frac{488}{45}x^6 - \frac{1998848}{45}t^6 + \frac{3904}{15}x^5t - \frac{7808}{3}x^4t^2 + \frac{124928}{9}x^3t^3 \\ & - \frac{124928}{3}x^2t^4 + \frac{999424}{15}xt^5 + \frac{1108}{63}x^8 - \frac{35456}{63}x^7t + \frac{70912}{9}x^6t^2 \\ & - \frac{567296}{9}x^5t^3 + \frac{2836480}{9}x^4t^4 - \frac{9076736}{9}x^3t^5 + \frac{18153472}{9}x^2t^6 \\ & - \frac{145227776}{63}xt^7 + \frac{72613888}{63}t^8 + \cdots . \end{aligned} \tag{7}$$

3 Basic ideas of homotopy perturbation method

In this letter we study time-dependent differential equations, so we present basic ideas of HPM in a way which best suits our examples. For a good understanding of the homotopy perturbation method the reader is referred to He's works [13, 14]. To describe the basic ideas, consider the time-dependent differential equation in the following general form

$$A(y(r,t)) - f(r,t) = 0, \tag{8}$$

where A is a differential operator, $y(r,t)$ is an unknown function, r and t denote spatial and temporal independent variables, respectively, and $f(r,t)$ is a known analytic function. A, generally speaking, can be divided into two parts, L and N,

$$A = L + N, \tag{9}$$

where L is a simple part which is easy to handle and N contains the remaining parts of A. Using homotopy technique one can construct a homotopy $\phi(r,t;q)$ satisfying

$$\begin{array}{l} H(\phi(r,t;q),q) = \\ \qquad (1-q)\{L(\phi(r,t;q)) - L(v_0(r,t))\} + q\{A(\phi(r,t;q)) - f(r,t)\} = 0 \quad , \end{array} \tag{10}$$

where $q \in [0,1]$ is an embedding parameter and $v_0(r,t)$ is an initial guess for Eq.(8). Eq.(10) is called *homotopy equation.* Equivalently it can be written as follows:

$$L(\phi(r,t;q)) - L(v_0(r,t)) + q\{N(\phi(r,t;q)) + L(v_0(r,t)) - f(r,t)\} = 0. \tag{11}$$

Clearly we have

$$q = 0 \quad \Rightarrow \quad H(\phi(r,t;0),0) = L(\phi(r,t;0)) - L(v_0(r,t)) = 0, \tag{12}$$

$$q = 1 \quad \Rightarrow \quad H(\phi(r,t;1),1) = A(\phi(r,t;1)) - f(r,t) = 0, \tag{13}$$

which the latter is actually Eq.(8) with solution $y(r,t)$. Eq.(12) has $v_0(r,t)$ as one of its solutions and in the case where L is assumed to be linear, $v_0(r,t)$ is the only solution. So we have

$$\phi(r,t;1) = y(r,t),$$

$$\phi(r,t;0) = v_0(r,t).$$

The changing process of q from zero to unity is just that of $\phi(r,t;q)$ from $v_0(r,t)$ to $y(r,t)$, this is called deformation. If the embedding parameter q ($0 \leq q \leq 1$) is considered as a "*small parameter*", applying the classic perturbation technique we can naturally assume that the solution to Eqs.(12) and (13) can be given as a power series in q, i.e.,

$$\phi(r,t;q) = u_0(r,t) + u_1(r,t)q + u_2(r,t)q^2 + \cdots . \tag{14}$$

Using (14) for $q = 1$, one has

$$y(r,t) = u_0(r,t) + u_1(r,t) + u_2(r,t) + \cdots . \tag{15}$$

which is the approximate solution to Eq.(8)(see, e.g., [13, 14]).

4 Application

For the aforementioned time-dependent differential equations, we propose to choose the operator L and the initial guess to be

$$L\phi = \frac{\partial \phi}{\partial t}, \quad v_0(x,t) = 0. \tag{16}$$

In this way we have an operator which is easy to handle and closely related to the original equation. On the other side choosing the initial guess to be the zero function, we escape from appearance of secular terms in the solution expression. Moreover in this way one has less difficulty in subsequently solving the resulted equations.

4.1 *Burgers equation*

According to Eq.(1) by choosing (16) the homotopy equation is

$$\phi_t + q\{\phi_x \phi - \phi_{xx}\} = 0. \tag{17}$$

Using (14) and (17), then equating the terms with equal powers of q, we have the following system of differential equations

$$\begin{aligned}
u_{0t} &= 0, & u_0(x,0) &= \tfrac{1}{2} - \tfrac{1}{2}\tanh(\tfrac{1}{4}x),\\
u_{1t} + u_0 u_{0x} - u_{0xx} &= 0, & u_1(x,0) &= 0,\\
u_{2t} + u_1 u_{0x} + u_0 u_{1x} - u_{1xx} &= 0, & u_2(x,0) &= 0,\\
u_{3t} + u_2 u_{0x} + u_0 u_{2x} + u_1 u_{1x} - u_{2xx} &= 0, & u_3(x,0) &= 0,\\
u_{4t} + u_3 u_{0x} + u_0 u_{3x} + u_1 u_{2x} + u_2 u_{1x} - u_{3xx} &= 0, & u_4(x,0) &= 0,\\
u_{5t} + u_4 u_{0x} + u_0 u_{4x} + u_1 u_{3x} + u_3 u_{1x} + u_2 u_{2x} - u_{4xx} &= 0, & u_5(x,0) &= 0,\\
&\vdots & &\vdots
\end{aligned}$$

Subsequently solving the above system we have

$$\begin{aligned}
u_0(x,t) &= \frac{1}{2} - \frac{1}{2}\tanh(\frac{x}{4}),\\
u_1(x,t) &= \frac{-t}{16}\{\tanh^2(\frac{x}{4}) - 1\},\\
u_2(x,t) &= \frac{-t^2}{128}\tanh(\frac{x}{4})\{\tanh^2(\frac{x}{4}) - 1\},\\
u_3(x,t) &= \frac{-t^3}{3072}\{3\tanh^4(\frac{x}{4}) - 4\tanh^2(\frac{x}{4}) + 1\},\\
u_4(x,t) &= \frac{-t^4}{24576}\{3\tanh^4(\frac{x}{4}) - 5\tanh^2(\frac{x}{4}) + 2\},\\
u_5(x,t) &= \frac{-t^5}{983040}\{15\tanh^6(\frac{x}{4}) - 30\tanh^4(\frac{x}{4}) + 17\tanh^2(\frac{x}{4}) - 2\},\\
&\vdots
\end{aligned}$$

Using a 6-term approximation according to (15) one has

$$\begin{aligned}
y(x,t) \cong\ & u_0+u_1+u_2+u_3+u_4+u_5 \\
=\ & \{\frac{-1}{65536}\tanh^6(\frac{x}{4})+\frac{1}{32768}\tanh^4(\frac{x}{4})-\frac{17}{983040}\tanh^2(\frac{x}{4}) \\
& +\frac{1}{491520}\}t^5-\{\frac{1}{8192}\tanh^5(\frac{x}{4})+\frac{5}{24576}\tanh^3(\frac{x}{4}) \\
& -\frac{1}{12288}\tanh(\frac{x}{4})\}t^4-\{\frac{1}{1024}\tanh^4(\frac{x}{4})+\frac{1}{768}\tanh^2(\frac{x}{4})-\frac{1}{3072}\}t^3 \\
& -\{\frac{1}{128}\tanh^3(\frac{x}{4})+\frac{1}{128}\tanh(\frac{x}{4})\}t^2+\{\frac{1}{16}-\frac{1}{16}\tanh^2(\frac{x}{4})\}t \\
& -\frac{1}{2}\tanh(\frac{x}{4})+\frac{1}{2}.
\end{aligned}$$

Expanding this approximation using Taylor expansion about $(x,t)=(0,0)$ we have

$$\begin{aligned}
y(x,t) \cong\ & \frac{1}{2}-\frac{1}{8}x+\frac{1}{16}t-\frac{1}{3072}t^3+\frac{1}{512}xt^2-\frac{1}{256}x^2t+\frac{1}{384}x^3+\frac{1}{491520}t^5 \\
& -\frac{1}{49152}xt^4+\frac{1}{12288}x^2t^3-\frac{1}{6144}x^3t^2+\frac{1}{6144}x^4t-\frac{1}{15360}x^5 \\
& -\frac{17}{1321205760}t^7+\frac{17}{94371840}xt^6-\frac{17}{15728640}x^2t^5+\frac{17}{4718592}x^3t^4 \\
& -\frac{17}{2359296}x^4t^3+\frac{17}{1966080}x^5t^2-\frac{17}{2949120}x^6t+\frac{17}{10321920}x^7+\cdots,
\end{aligned}$$

which is exactly the Taylor expansion of the exact solution (3).

4.2 *RLW equation*

Consider the RLW equation (Eq.(4)). We apply HPM by choosing (16) so the homotopy equation is

$$\phi_t+q\{\phi_x+\phi_x\phi-\phi_{xxt}\}=0. \tag{18}$$

Using (14) and (18), then equating the terms with equal powers of q, we have the following system of differential equations

$$\begin{aligned}
u_{0t}=0,&\quad u_0(x,0)= 3\alpha\ \mathrm{sech}(\beta x),\\
u_{1t}+u_{0x}+u_0u_{0x}-u_{0xxt}=0,&\quad u_1(x,0)=0,\\
u_{2t}+u_{1x}+u_1u_{0x}+u_0u_{1x}-u_{1xxt}=0,&\quad u_2(x,0)=0,\\
u_{3t}+u_{2x}+u_2u_{0x}+u_0u_{2x}+u_1u_{1x}-u_{2xxt}=0,&\quad u_3(x,0)=0,\\
u_{4t}+u_{3x}+u_3u_{0x}+u_0u_{3x}+u_1u_{2x}+u_2u_{1x}-u_{3xxt}=0,&\quad u_4(x,0)=0,\\
\vdots\qquad&\qquad\vdots
\end{aligned}$$

Subsequently solving the above system we have

$$\begin{aligned}
u_0(x,t) =& \ 3\alpha \ \mathrm{sech}(\beta x),\\
u_1(x,t) =& \ \frac{6\alpha\beta\sinh(\beta x)(\cosh^2(\beta x)+3\alpha)}{\cosh^5(\beta x)}t,\\
u_2(x,t) =& \ \frac{3t\alpha\beta^2}{\cosh^8(\beta x)}\{2t\cosh^6(\beta x) - 3t\cosh^4(\beta x) + 24\alpha t\cosh^4(\beta x)\\
& -30t\alpha\cosh^2(\beta x) + 54t\alpha^2\cosh^2(\beta x) - 63t\alpha^2\\
& +8\beta\sinh(\beta x)\cosh^5(\beta x) - 24\beta\sinh(\beta x)\cosh^3(\beta x)\\
& +96\alpha\beta\sinh(\beta x)\cosh^3(\beta x) - 180\alpha\beta\sinh(\beta x)\cosh(\beta x)\},\\
&\vdots
\end{aligned}$$

Expanding the 6-term approximation $app_5(x,t) = \sum_{i=0}^{i=5} u_i(x,t)$ using Taylor expansion about $(x,t)=(0,0)$ and setting $\alpha = 0.02$ (as in [28]) gives

$$\begin{aligned}
app_5(x,t) \cong& \ 0.06 - 0.000294117647\,x^2 - 0.000305834955\,t^2\\
& +0.0006000189934\,xt + 8.569985394\times 10^{-15}\,x^2t\\
& -1.8\times 10^{-15}\,x^3 + 0.0000009611687808\,x^4\\
& -0.000003922889318\,x^3t + 0.000005974024045\,x^2t^2\\
& -0.000004199282705\,t^3x + 0.0000008984094719\,t^4 + \cdots .
\end{aligned}$$

On the other hand, if we expand the exact solution to Eq.(4) using Taylor expansion about $(x,t)=(0,0)$ (and again setting $\alpha = 0.02$) we have

$$\begin{aligned}
y(x,t) &= 3\alpha \ \mathrm{sech}(\beta(x-(1+\alpha)t))\\
&\cong 0.06 - 0.0003059999999\,t^2 + 0.0006\,xt - 0.0002941176470\,x^2\\
&\quad + 0.000001040399999\,t^4 - 0.000004079999998\,t^3x\\
&\quad + 0.000005999999997\,x^2t^2 - 0.000003921568625\,x^3t\\
&\quad + 0.0000009611687807\,x^4 + \cdots ,
\end{aligned}$$

which is very close to our 6-term approximation ($app5(x,t)$). The absolute value of error function $error(x,t) = app_5(x,t) - u_{exact}(x,t)$ at some points is tabulated in Table 1.

4.3 *modified KdV equation*

We consider the modified Korteweg-de Vries equation (5), applying HPM to this equation by choosing (16) one has the homotopy equation as follows:

$$\phi_t + q\{6\phi^2\phi_x + \phi_{xxx}\} = 0. \tag{19}$$

Using (14) and (19), then equating the terms with equal powers of q, we have the following system of differential equations

$$\begin{aligned}
u_{0t} &= 0, \quad u_0(x,0) = 2\operatorname{sech}(2x),\\
u_{1t} + 6u_{0x}u_0^2 + u_{0xxx} &= 0, \quad u_1(x,0) = 0,\\
u_{2t} + 6u_{1x}u_0^2 + 12u_1u_{0x}u_0 + u_{1xxx} &= 0, \quad u_2(x,0) = 0,\\
u_{3t} + 12u_{1x}u_0u_1 + 6(u_1^2 + 2u_0u_2)u_{0x} + 6u_0^2u_{2x} + u_{2xxx} &= 0, \quad u_3(x,0) = 0,\\
&\vdots \qquad\qquad \vdots
\end{aligned}$$

Subsequently solving the above system we have

$$\begin{aligned}
u_0(x,t) &= 2\operatorname{sech}(2x),\\
u_1(x,t) &= \frac{16t\sinh(2x)}{\cosh^2(2x)},\\
u_2(x,t) &= \frac{64t^2(\cosh^2(2x)-2)}{\cosh^3(2x)},\\
u_3(x,t) &= \frac{1024t^4(\cosh^4(2x) - 2\cosh^2(2x) + 24)}{3\cosh^5(2x)},\\
&\vdots
\end{aligned}$$

Expanding the 6-term approximation $app_5(x,t) = \sum_{i=0}^{i=5} u_i(x,t)$ using Taylor expansion about $(x,t) = (0,0)$ gives

$$\begin{aligned}
app_5(x,t) \cong\ & 2 - 4x^2 - 64t^2 + 32xt - \frac{5120}{3}xt^3 + \frac{5120}{3}t^4 + 640x^2t^2\\
& - \frac{320}{3}x^3t + \frac{20}{3}x^4 - \frac{124928}{3}x^2t^4 - \frac{7808}{3}x^4t^2 + \frac{3904}{15}x^5t\\
& - \frac{488}{45}x^6 + \frac{124928}{9}x^3t^3 + \frac{999424}{15}xt^5 + \frac{2836480}{9}x^4t^4\\
& - \frac{567296}{9}x^5t^3 + \frac{70912}{9}x^6t^2 - \frac{9076736}{9}x^3t^5 - \frac{35456}{63}x^7t\\
& + \frac{1108}{63}x^8 + \cdots .
\end{aligned} \tag{20}$$

On the other hand, if we truncate (7), the Taylor expansion of the exact solution (6), up to t^5 we would have (20). We should use the terms up to power 5 with respect to t, because the greatest power of t in our approximation, $app_5(x,t)$, is 5. If one wants to obtain more accurate solutions, he/she should calculate more terms according to (15) to achieve exactly (7).

5 Conclusions

In the present work, we solved some nonlinear time-dependent differential equations using HPM. We considered Burgers equation, regularized long wave equation and modified Korteweg-de Vries equation. When applying HPM we used he same structure to handle these equations and we obtained the Taylor expansion of the exact solution. The results show the efficiency of HPM in solving nonlinear equations.

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Table 1 -Absolute errors of a 6-term approximation for RLW

$t_i \diagdown x_i$	2	-2	4	-4	6	-6
0.1	3.40E-9	1.92E-9	1.09E-9	2.56E-9	1.65E-9	5.1E-10
0.2	7.03E-9	1.31E-9	3.2E-10	5.42E-9	5.54E-9	2.85E-9
0.3	9.49E-9	2.64E-9	5.21E-9	7.4E-9	1.153E-8	6.49E-9
0.4	9.23E-9	1.048E-8	1.436E-8	7.44E-9	1.931E-8	1.078E-8
0.5	6.62E-9	2.242E-8	2.847E-8	4.35E-9	2.822E-8	1.49E-8
1.0	1.439E-7	1.309E-7	1.773E-7	8.869E-8	4.61E-8	2.55E-9
2.0	1.537E-6	2.576E-7	2.379E-7	6.001E-7	1.156E-6	6.322E-7
3.0	3.123E-6	5.908E-6	4.867E-6	1.048E-6	8.688E-6	1.928E-6
4.0	4.429E-6	2.734E-5	3.013E-5	1.465E-5	3.297E-5	3.741E-7
5.0	4.68E-5	8.212E-5	1.071E-4	6.208E-5	9.164E-5	1.97E-5

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