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Spline Collocation Approach for the Numerical Solution of a Generalized System of Second-Order Boundary-Value Problems

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Abstract

In this paper, a spline finite element approach is manipulated for the numerical solution of an extended system of second-order boundary-value problems. The efficiency of the proposed method is examined by comparing the results with the existing exact closed form solution. The numerical results demonstrate that the method is efficient and quite accurate when contrasted with other methods and required relatively less computational work.

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1 Introduction

In this paper, we study a generalized nonlinear system of second-order boundary-value problems given by

$$\begin{aligned} a_0(x)u'' + a_1(x)u' + a_2(x)u + a_3(x)v'' + a_4(x)v' + a_5(x)v + a_6(x)u'v' + g_1(x, u, v) = f_1(x) \\ b_0(x)u'' + b_1(x)u' + b_2(x)u + b_3(x)v'' + b_4(x)v' + b_5(x)v + b_6(x)u'v' + g_2(x, u, v) = f_2(x) \end{aligned} \quad (1)$$

where $a \le x \le b$. The nonlinear system is subject to the following specified boundary conditions:

$$\begin{array}{ll} u(a) = \eta_1 & u(b) = \eta_2 \\ v(a) = \mu_1 & v(b) = \mu_2 \end{array} \tag{2}$$

where η_1, η_2, μ_1 and μ_2 are constants. Also, g_1, g_2 are nonlinear functions in u and v, $u, v \in W_2^3[a,b]$, $f_i - g_i \in W_2^1[a,b]$, $i = 1, 2$. The functions $a_j(x)$ and $b_j(x)$ are continuous for $j = 0, 1, 2, ..., 6$.

In recent years, nonlinear equations [4] as well as nonlinear systems of boundary value problems have been extensively investigated in the literature due to their wide range of applicability in engineering and other disciplines. Special cases of system (1)-(2), namely the one which does not include the $u'v'$-term, has been explored by a number of authors. Obtaining exact and/or numerical solutions of nonlinear systems, particulary the systems which arise in applications, have been the focus of attention. Geng and Cui [5] presented a method to obtain the analytical and approximate solutions of linear and nonlinear system of second-order boundary value problems. The analytical solution is represented in the form of series in the reproducing kernel space. Cheng and Zhong [3] discussed the existence of solutions to second order systems. Valanarasu and Ramanujam [10] suggested a method for solving a system of singularly perturbed second-order ordinary differential equations. In [7] Dehghan and Saadatmandi employed a numerical method based on Sinc-collocation method for the solution of second-order nonlinear systems. Saadatmandi and J. Askari [8] solved similar systems by using the Chebyshev finite element method. Lu [6] introduced a variational iteration approach to solve systems analogous to problem (1)-(2). In [9] Saadatmandi et al. proposed a homotopy perturbation method for solving a class of non-linear systems of second-order boundary-value problems. The method yields solutions in convergent series forms with easily computable terms and the technique does not require any discretization, linearization or small perturbations. For further approaches to tackle nonlinear systems we refer the reader to the references in [9].

In this paper, we will apply a finite element collocation approach, based on cubic splines, to obtain a solution to the wide class of nonlinear systems of boundary value problems given in (1)-(2). It is worth noting that the system we are studying is more general than the ones discussed in the above mentioned references as it includes the extra nonlinear $u'v'$-term. The spline finite element approach is widely utilized (see [1] and [2]) for the numerical solution of nonlinear problems arising in real world applications. A number of numerical examples are examined to illustrate the applicability and efficiency of the finite element scheme. Comparison is made between the exact analytical solution and the numerical solution obtained by the spline collocation approach. The results indicate that the current approach is convenient and yields accurate results using only a few number mesh points.

The structure of this paper is organized as follows: in Section 2, we describe the cubic B-spline collocation approach for the numerical solution of the system of the second-order boundary-value problem. In section 3, the method is implemented on a number of examples of nonlinear systems using different choices of the number of mesh points. The numerical results are compared with the exact solutions and a conclusion is given that summarizes the outcomes of the simulations.

2 Finite Element Method

In this section, we present the collocation approach using cubic splines is presented for the numerical solution of the extended class of nonlinear systems of second-order boundary-value problems given in (1)-(2). We will seek a finite-element solution for solving the nonlinear system of boundary-value problems. To construct such an approximate solution, we consider the nodal points x_i on the interval $[a, b]$ where

$$a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

Note that if the nodal points are equidistant from each other, then we have $x_i = a + ih, \;\; i = 0, 1, 2, ..., n$ where $h = \dfrac{b-a}{n}$ on the interval $[a, b]$. Let $\psi(x)$ and $\phi(x)$ be shape functions that satisfy the boundary conditions (2) and are written as a linear combination of $n + 3$ shape functions given by

$$\psi(x) = \sum_{i=-3}^{n-1} \alpha_i \psi_i(x) \qquad \text{and} \qquad \phi(x) = \sum_{i=-3}^{n-1} \beta_i \psi_i(x) \tag{3}$$

The α_i's and β_i's are unknown real coefficients and the $\psi_i(x)$ are the cubic B-splines functions defined as follows:

$$\psi_i(x) = \frac{1}{h^3} \begin{cases} (x - x_i)^3, & [x_i, x_{i+1}] \\ h^3 + 3h^2(x - x_{i+1}) + 3h(x - x_{i+1})^2 - 3(x - x_{i+1})^3, & [x_{i+1}, x_{i+2}] \\ h^3 + 3h^2(x_{i+3} - x) + 3h(x_{i+3} - x)^2 - 3(x_{i+3} - x)^3, & [x_{i+2}, x_{i+3}] \\ (x_{i+4} - x)^3, & [x_{i+3}, x_{i+4}] \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

where $h = x_{i+1} - x_i$. In particular, since $x_0 = 0$ then $\psi_0(x)$ is given by

$$\psi_0(x) = \frac{1}{h^3}\begin{cases} x^3, & [0, h] \\ h^3 + 3h^2(x-h) + 3h(x-h)^2 - 3(x-h)^3, & [h, 2h] \\ h^3 + 3h^2(3h-x) + 3h(3h-x)^2 - 3(3h-x)^3, & [2h, 3h] \\ (4h-x)^3, & [3h, 4h] \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

From (4), the values of ψ_i, ψ_i' and ψ_i'' at the nodal points $t_i = ih$ are given according to the following table:

Nodes	ψ_i	ψ_i'	ψ_i''
x_i	0	0	0
x_{i+1}	1	$\frac{3}{h}$	$\frac{6}{h^2}$
x_{i+2}	4	0	$-\frac{12}{h^2}$
x_{i+3}	1	$-\frac{3}{h}$	$\frac{6}{h^2}$
x_{i+4}	0	0	0

Table 1. ψ_i, ψ_i', and ψ_i'' evaluated at the nodal points.

Next, we will present the finite-element collocation approach for approximating the solution of the generalized nonlinear system of boundary-value problems. We assume that the solution u is approximated by $\psi(x)$ while the second solution v is estimated by $\phi(x)$. Substituting the approximate solutions (3) into equations (1) yields

$$\sum_{i=-3}^{n-1} \alpha_i \left[a_0(x_j)\psi_i''(x_j) + a_1(x_j)\psi_i'(x_j) + a_2(x_j)\psi_i(x_j)\right] + \sum_{i=-3}^{n-1} \beta_i \left[a_3(x_j)\psi_i''(x_j) + a_4(x_j)\psi_i'(x_j) + a_5(x_j)\psi_i(x_j)\right] + a_6(x_j)\left(\sum_{i=-3}^{n-1} \alpha_i\psi_i'(x_j)\right)\left(\sum_{i=-3}^{n-1} \beta_i\psi_i'(x_j)\right) + g_1\left(x_j, \sum_{i=-3}^{n-1} \alpha_i\psi_i(x_j), \sum_{i=-3}^{n-1} \beta_i\psi_i(x_j)\right) = f_1(x_j)$$

$$j = 0, 1, 2, ..., n \tag{6}$$

and for the second differential equation we obtain

$$\sum_{i=-3}^{n-1} \alpha_i \left[b_0(x_j)\psi_i''(x_j) + b_1(x_j)\psi_i'(x_j) + b_2(x_j)\psi_i(x_j)\right] + \sum_{i=-3}^{n-1} \beta_i \left[b_3(x_j)\psi_i''(x_j) + b_4(x_j)\psi_i'(x_j) + b_5(x_j)\psi_i(x_j)\right] + b_6(x_j)\left(\sum_{i=-3}^{n-1} \alpha_i\psi_i'(x_j)\right)\left(\sum_{i=-3}^{n-1} \beta_i\psi_i'(x_j)\right) + g_2\left(x_j,\ \sum_{i=-3}^{n-1} \alpha_i\psi_i(x_j),\ \sum_{i=-3}^{n-1} \beta_i\psi_i(x_j)\right) = f_2(x_j)$$

$$j = 0, 1, 2, ..., n \tag{7}$$

The above system consists of $2n + 2$ equations in $2n + 6$ unknowns. The boundary conditions in (2) give the following four equations:

For $u(a) = \eta_1$ and $u(b) = \eta_1$ we have, respectively,

$$u(a) = \sum_{i=-3}^{n-1} \alpha_i\psi_i(t_j) = \eta_1, \qquad j = 0 \tag{8}$$

$$u(b) = \sum_{i=-3}^{n-1} \alpha_i\psi_i(x_j) = \eta_2, \qquad j = n \tag{9}$$

For $v(a) = \mu_1$ and $v(b) = \mu_2$ we have, respectively,

$$v(a) = \sum_{i=-3}^{n-1} \beta_i\psi_i(t_j) = \mu_1, \qquad j = 0 \tag{10}$$

$$v(b) = \sum_{i=-3}^{n-1} \beta_i\psi_i(x_j) = \mu_2, \qquad j = n \tag{11}$$

The values of $\psi_i(x_j), \psi_i'(x_j)$ and $\psi_i''(x_j)$ at the nodal points $x_j,\ j = 0, 1, ..., n$ are determined from Table 1.

The system of equations in (6), (8), and (9) can be written in matrix form as follows:

$$\mathbf{C_1}\ \mathbf{d} + \mathbf{M_1}\ \mathbf{e} + \mathbf{v_1} + \mathbf{g_1} = \mathbf{f_1} \tag{12}$$

where

$$\mathbf{C_1} = \begin{bmatrix} 1 & 4 & 1 & 0 & 0 & ... & 0 \\ r_0 & s_0 & p_0 & 0 & 0 & ... & 0 \\ 0 & r_1 & s_1 & p_1 & 0 & ... & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & ... & r_n & s_n & p_n \\ 0 & 0 & 0 & ... & 1 & 4 & 1 \end{bmatrix}$$

We have

$$r_j = \frac{6a_{0j}}{h^2} - \frac{3a_{1j}}{h} + a_{2j}, \quad s_j = -\frac{12a_{0j}}{h^2} + 4a_{2j}, \quad p_j = \frac{6a_{0j}}{h^2} + \frac{3a_{1j}}{h} + a_{2j}, \quad j = 0, 1, ...n$$

given that

$$a_{0j} = a_0(x_j), \quad a_{1j} = a_1(x_j), \quad a_{2j} = a_2(x_j) \quad \text{where} \quad x_j = a + jh$$

and

$$\mathbf{M_1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & ... & 0 \\ q_0 & w_0 & z_0 & 0 & 0 & ... & 0 \\ 0 & q_1 & w_1 & z_1 & 0 & ... & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & ... & q_n & w_n & z_n \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \end{bmatrix}$$

We have

$$q_j = \frac{6a_{3j}}{h^2} - \frac{3a_{4j}}{h} + a_{5j}, \quad w_j = -\frac{12a_{3j}}{h^2} + 4a_{5j}, \quad z_j = \frac{6a_{3j}}{h^2} + \frac{3a_{4j}}{h} + a_{5j}, \quad j = 0, 1, ...n$$

given that

$$a_{3j} = a_3(x_j), \quad a_{4j} = a_4(x_j), \quad a_{5j} = a_5(x_j) \quad \text{where} \quad x_j = a + jh$$

$$\mathbf{g_1} = \begin{bmatrix} 0 \\ g_1\left(\alpha_{-3}+4\alpha_{-2}+\alpha_{-1},\ \beta_{-3}+4\beta_{-2}+\beta_{-1}\right) \\ g_1\left(\alpha_{-2}+4\alpha_{-1}+\alpha_{0},\ \beta_{-2}+4\beta_{-1}+\beta_{0}\right) \\ g_1\left(\alpha_{-1}+4\alpha_{0}+\alpha_{1},\ \beta_{-1}+4\beta_{0}+\beta_{1}\right) \\ . \\ . \\ . \\ g_1\left(\alpha_{n-4}+4\alpha_{n-3}+\alpha_{n-2},\ \beta_{n-4}+4\beta_{n-3}+\beta_{n-2}\right) \\ g_1\left(\alpha_{n-3}+4\alpha_{n-2}+\alpha_{n-1},\ \beta_{n-3}+4\beta_{n-2}+\beta_{n-1}\right) \\ 0 \end{bmatrix}$$

$$\mathbf{v_1} = \begin{bmatrix} 0 \\ a_6(x_0)\left(\sum_{i=-3}^{n-1}\alpha_i\psi_i'(x_0)\right)\left(\sum_{i=-3}^{n-1}\beta_i\psi_i'(x_0)\right) \\ a_6(x_1)\left(\sum_{i=-3}^{n-1}\alpha_i\psi_i'(x_1)\right)\left(\sum_{i=-3}^{n-1}\beta_i\psi_i'(x_1)\right) \\ . \\ . \\ . \\ a_6(x_{n-1})\left(\sum_{i=-3}^{n-1}\alpha_i\psi_i'(x_{n-1})\right)\left(\sum_{i=-3}^{n-1}\beta_i\psi_i'(x_{n-1})\right) \\ a_6(x_n)\left(\sum_{i=-3}^{n-1}\alpha_i\psi_i'(x_n)\right)\left(\sum_{i=-3}^{n-1}\beta_i\psi_i'(x_n)\right) \\ 0 \end{bmatrix}$$

$$\mathbf{d^T} = \begin{bmatrix} \alpha_{-3} & \alpha_{-2} & \alpha_{-1} & \alpha_0 & ... & \alpha_{n-3} & \alpha_{n-2} & \alpha_{n-1} \end{bmatrix}$$

$$\mathbf{e^T} = \begin{bmatrix} \beta_{-3} & \beta_{-2} & \beta_{-1} & \beta_0 & ... & \beta_{n-3} & \beta_{n-2} & \beta_{n-1} \end{bmatrix}$$

and

$$\mathbf{f_1^T} = \begin{bmatrix} \eta_1 & f_1(x_0) & f_1(x_1) & f_1(x_2) & ... & f_1(x_{n-1}) & f_1(x_n) & \eta_2 \end{bmatrix}$$

In a similar fashion, the system of equations in (7), (10), and (11) can be written in matrix form as follows:

$$\mathbf{C_2}\,\mathbf{d} + \mathbf{M_2}\,\mathbf{e} + \mathbf{v_2} + \mathbf{g_2} = \mathbf{f_2} \tag{13}$$

The matrices $\mathbf{C_2}, \mathbf{M_2}, \mathbf{v_2}, \mathbf{g_2}$ and $\mathbf{f_2}$ are similar to $\mathbf{C_1}, \mathbf{M_1}, \mathbf{v_1}, \mathbf{g_1}$ and $\mathbf{f_1}$ except that the functions $a_i, i = 0, 1, ..., 6$ and f_1 are replaced by the functions $b_i, i = 0, 1, ..., 5, 6$ and f_2, respectively. As for the boundary conditions, we replace η_1, η_2 in $\mathbf{f_1}$ by μ_1, μ_2 to obtain $\mathbf{f_2}$.

Systems (12) and (13) can be combined as one system as follows:

$$\begin{bmatrix} \mathbf{C_1} \\ \mathbf{C_2} \end{bmatrix} \mathbf{d} + \begin{bmatrix} \mathbf{M_1} \\ \mathbf{M_2} \end{bmatrix} \mathbf{e} + \begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{bmatrix} + \begin{bmatrix} \mathbf{g_1} \\ \mathbf{g_2} \end{bmatrix} = \begin{bmatrix} \mathbf{f_1} \\ \mathbf{f_2} \end{bmatrix} \tag{14}$$

The system of equations given in (14) is solved using the computer algebra system *Maple-11*.

3 Numerical Examples and Conclusion

In this section, we show a number of numerical simulations of our model nonlinear system of second-order boundary-value problem which were produced by using the spline collocation approach. The aim of these simulations is the validation of the numerical solution and illustration of the accuracy of the proposed method. We now demonstrate and test the practicality and usefulness of the collocation approach with five numerical examples. Further, the absolute errors in the analytical solutions are calculated. All computations were carried out using *Maple-11*.

Example 1. We will consider the following special case of the second-order system of boundary-value problems (1)-(2).

$$\begin{cases} u''(x) + u'(x) + xu(x) + v'(x) + 2xv(x) = f_1(x) \\ v''(x) + v(x) + 2u'(x) + x^2 u(x) = f_2(x) \\ u(0) = u(1) = 0, \quad v(0) = v(1) = 0 \end{cases} \tag{15}$$

where $0 \le x \le 1$, $f_1(x) = -2(1+x)\cos x + \pi \cos \pi x + 2x \sin \pi x + (4x - 2x^2 - 4)\sin x$ and $f_2(x) = -4(x-1)\cos x - 2(2 - x^2 + x^3)\sin x - (\pi^2 - 1)\sin \pi x$. Problem (15) has the exact solutions $u(x) = 2(1-x)\sin x$ and $v(x) = \sin \pi x$.

In Table 1, the numerical solution obtained by the B-spline collocation method, using $n = 10$ nodal points, at the mesh points $x = 0.1, 0.2, ...0.9$ for problem (15) is compared with the exact solution. The observed absolute errors between the exact solution and that obtained by the spline collocation method at various values of mesh points are given. This example is taken form reference [5], though it is worth pointing out that there is a misprint in the plus sign in the value of $f_2(x)$.

x	Spline Solution $u(x)$	|Error|	Spline Solution $v(x)$	|Error|
0.0	0.0	0.0	0.0	0.0
0.1	0.1787131430	9.9×10^{-4}	0.3061130050	2.9×10^{-3}
0.2	0.3163211912	1.5×10^{-3}	0.5824083004	5.4×10^{-3}
0.3	0.4119666444	1.8×10^{-3}	0.8017688743	7.2×10^{-3}
0.4	0.4656031566	1.7×10^{-3}	0.9426730925	8.4×10^{-3}
0.5	0.4779849620	1.4×10^{-3}	0.9912961778	8.7×10^{-3}
0.6	0.4506457864	1.1×10^{-3}	0.9428594398	8.2×10^{-3}
0.7	0.3858674368	6.6×10^{-4}	0.8020951548	6.9×10^{-3}
0.8	0.2866383882	3.0×10^{-4}	0.5827815507	5.0×10^{-3}
0.9	0.1566027428	6.3×10^{-5}	0.3063933657	2.6×10^{-3}
1.0	0.0	0.0	0.0	0.0

Table 1. Numerical solution of system (3.15) using 10 nodal points.

Example 2. We now consider a second nonlinear system which includes the nonlinear $u'v'$-term.

$$\begin{cases} (x-2)u''(x) + u'(x)v'(x) - u(x)v^2(x) = f_1(x) \\ v''(x) - u''(x) + xu'(x)v'(x) + xv(x) - u^2(x) - v^2(x) = f_2(x) \\ u(0) = 0,\ u(1) = 1, \quad v(0) = v(1) = 0 \end{cases} \tag{16}$$

where $0 \leq x \leq 1$, $f_1(x) = \cos^3 \pi x + (3x-1)\cos^2 \pi x - (\pi^2 \sin \pi x + \pi^2 x - 2\pi^2 - 3\pi + 1)\cos \pi x - \pi^2 x \sin \pi x - 3x + 1$ and $f_2(x) = (2 + \pi^2 - 6x + 4\pi x - \pi^2 x \sin \pi x)\cos \pi x - \pi^2 \sin \pi x - 9x^2 + 6x - 2$. System (16) has the exact solutions $u(x) = 3x - 1 + \cos \pi x$ and $v(x) = \sin \pi x$.

In Table 2, the numerical solution obtained by the B-spline collocation method, using $n = 10$ nodal points, at the mesh points $x = 0.1, 0.2, ...0.9$ for system (16) is compared with the exact solution by giving the absolute error.

x	Spline Solution $u(x)$	|Error|	Spline Solution $v(x)$	|Error|
0.0	0.0	0.0	0.0	0.0
0.1	0.2502396371	8.2×10^{-4}	0.3072520035	1.8×10^{-3}
0.2	0.4078584554	1.2×10^{-3}	0.5842589692	3.5×10^{-3}
0.3	0.4868145178	9.7×10^{-4}	0.8040962329	4.9×10^{-3}
0.4	0.5087270719	2.9×10^{-4}	0.9453507046	5.7×10^{-3}
0.5	0.5006947233	6.9×10^{-4}	0.9942112773	5.8×10^{-3}
0.6	0.4926459573	1.7×10^{-3}	0.9458537599	5.2×10^{-3}
0.7	0.5144908241	2.3×10^{-3}	0.8049538009	4.1×10^{-3}
0.8	0.5932906556	2.3×10^{-3}	0.5852368223	2.5×10^{-3}
0.9	0.7506158228	1.7×10^{-3}	0.3080750888	9.4×10^{-4}
1.0	1.0	0.0	0.0	0.0

Table 2. Numerical solution of system (3.16) using 10 nodal points.

Example 3. Analogous to example 2, we again consider a nonlinear system that includes the nonlinear $u'v'$-term.

$$\begin{cases} 3u''(x) + 3xu(x) + 3v(x) - u'(x)v'(x) = 8 - x \\ v''(x) + x^3u'(x) + x^2u(x) - \frac{x}{2}u'(x)v'(x) = x^3 + x^2 + 7x \\ u(0) = 0,\ u(1) = 3, \quad v(0) = v(1) = 0 \end{cases} \tag{17}$$

where $0 \leq x \leq 1$. System (17) has the exact solutions $u(x) = x^2 + 2x$ and $v(x) = x^3 - x$.

In Table 3, we give the absolute error between the exact solution of system (17) and the numerical results obtained by the spline collocation method. It is evident from the table that the absolute errors are extremely small using only 5 nodal points at the mesh points $x = 0.2, 0.4, 0.6, 0.8$. We observe that the error is too small as expected and that is due to the fact that the exact solution is a polynomial.

x	Spline Solution $u(x)$	|Error|	Spline Solution $v(x)$	|Error|
0.0	0.0	0.0	0.0	0.0
0.2	0.439999962	3.8×10^{-8}	−0.1920000231	2.3×10^{-8}
0.4	0.959999947	5.3×10^{-8}	−0.3360000430	4.3×10^{-8}
0.6	1.559999937	6.3×10^{-8}	−0.3840000618	6.2×10^{-8}
0.8	2.239999932	6.8×10^{-8}	−0.2880000764	7.6×10^{-8}
1.0	3.0	0.0	0.0	0.0

Table 3. Numerical solution of system (3.17) using 5 nodal points.

Example 4. We will now consider the following system:

$$\begin{cases} u''(x) - xv'(x) + u(x) = x^3 - 2x^2 + 6x \\ v''(x) + xu'(x) + u(x)v(x) = x^5 - x^4 + 2x^3 + x^2 - x + 2 \\ u(0) = u(1) = 0, \quad v(0) = v(1) = 0 \end{cases} \tag{18}$$

where $0 \leq x \leq 1$. System (18) has the exact solutions $u(x) = x^3 - x$ and $v(x) = x^2 - x$ (see [9]).

In Table 4, the numerical solution obtained by the B-spline collocation method, using $n = 5$ nodal points, at the mesh points $x = 0.2, 0.4, 0.6, 0.8$ for system (18) is compared with the exact solution. Note that 5 nodal points suffice to obtain a very accurate numerical solution using spline collocation. The absolute error is extremely small and this is again due to the fact that the exact solution is a polynomial.

x	Spline Solution $u(x)$	|Error|	Spline Solution $v(x)$	|Error|
0.0	0.0	0.0	0.0	0.0
0.2	-0.192000000	0.0	-0.1600000022	2.2×10^{-9}
0.4	-0.336000000	0.0	-0.2400000043	4.3×10^{-9}
0.6	-0.384000003	3.0×10^{-10}	-0.2400000054	5.4×10^{-9}
0.8	-0.288000003	3.0×10^{-10}	-0.1600000039	3.9×10^{-9}
1.0	0.0	0.0	0.0	0.0

Table 4. Numerical solution of system (3.18) using 5 nodal points.

Example 5. Finally, we explore the following nonlinear system:

$$\begin{cases} u''(x) + xv(x) + xu^2(x) = f_1(x) \\ v''(x) + xu'(x) + v(x) = f_2(x) \\ u(0) = u(1) = 0, \quad v(0) = v(1) = 0 \end{cases} \tag{19}$$

where $0 \leq x \leq 1$, $f_1(x) = x\sin^2 \pi x - \pi^2 \sin \pi x + x^4 - 3x^3 + 2x^2$ and $f_2(x) = \pi x \cos \pi x + x^3 - 3x^2 + 8x - 6$. System (19) has the exact solutions (see [9]) $u(x) = \sin \pi x$ and $v(x) = x^3 - 3x^2 + 2x$.

In Table 5, we give the absolute error between the exact solution for system (19) and the numerical solution obtained by the spline collocation method using $n = 5$ nodal points, at the mesh points $x = 0.2, 0.4, 0.6, 0.8$.

x	Spline Solution $u(x)$	|Error|	Spline Solution $v(x)$	|Error|
0.0	0.0	0.0	0.0	0.0
0.2	0.5674249067	2.0×10^{-2}	0.2883629969	3.6×10^{-4}
0.4	0.9178594327	3.3×10^{-2}	0.3852886128	1.3×10^{-3}
0.6	0.9176526443	3.3×10^{-2}	0.3385285446	2.5×10^{-3}
0.8	0.5671190563	2.0×10^{-2}	0.1947324189	2.7×10^{-3}
1.0	0.0	0.0	0.0	0.0

Table 5. Numerical solution of system (3.19) using 5 nodal points.

In Table 6, we give the absolute error between the exact solution and the numerical solution for the same system (19) using $n = 10$ nodal points instead of 5 points, at the mesh points $x = 0.1, 0.2, ..., 0.9$. It is evident from Tables 5 and 6 that as we increase the number of mesh points, the proposed methodology lead to higher accuracy.

x	Spline Solution $u(x)$	|Error|	Spline Solution $v(x)$	|Error|
0.0	0.0	0.0	0.0	0.0
0.1	0.3063019976	2.7×10^{-3}	0.1710196220	2.0×10^{-5}
0.2	0.5826059129	5.2×10^{-3}	0.2880642272	6.4×10^{-5}
0.3	0.8018581535	7.2×10^{-3}	0.3571512150	1.5×10^{-4}
0.4	0.9426015341	8.5×10^{-3}	0.3842837716	2.8×10^{-4}
0.5	0.9910728566	8.9×10^{-3}	0.3754462747	4.5×10^{-4}
0.6	0.9425421010	8.5×10^{-3}	0.3366028703	6.0×10^{-4}
0.7	0.8017665493	7.3×10^{-3}	0.2737000947	7.0×10^{-4}
0.8	0.5825209680	5.3×10^{-3}	0.1926736741	6.7×10^{-4}
0.9	0.3062536908	2.8×10^{-3}	0.0994585370	4.6×10^{-4}
1.0	0.0	0.0	0.0	0.0

Table 6. Numerical solution of system (3.19) using 10 nodal points.

In conclusion, the cubic spline finite element approach is suitable for obtaining numerical solutions for the extended system of second order boundary value problems given in (1)-(2). The numerical results show accuracy of the approach compared with other existing methods and also confirm that the accuracy is improved if we double the number of mesh points. In particular, the approach is simple and efficient and can be extended to other classes of systems of boundary value problems.

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