

# On Global Existence of Mild Solutions of Second Order Volterra Integrodifferential Equations

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## Abstract

In this paper, we study the existence and uniqueness of mild solutions for more general second order initial value problems, with nonlocal conditions, by using the Banach fixed point theorem and the theory of strongly continuous cosine family.

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## 1 Introduction

Let  $X$  be a Banach space with norm  $\|\cdot\|$  and throughout this paper we assume the notation  $J = [0, b]$ . Let  $B = C(J; X)$  be Banach space of all continuous functions from  $J$  into  $X$ , endowed with the norm

$$\|x\|_b = \sup\{\|x(t)\| : x \in B\}, \quad t \in J.$$

In the present paper we consider the following second order nonlinear integrodifferential equations with nonlocal conditions:

$$\frac{d}{dt}[x'(t) + g(t, x(t))] = Ax(t) + f\left(t, x(t), \int_0^t k(t, s, x(s))ds\right), \quad t \in J, \quad (1)$$

$$x(0) = x_0 + q(x), \quad (2)$$

$$x'(0) = y_0 + p(x), \quad (3)$$

where  $A$  is an infinitesimal generator of a strongly continuous cosine family  $\{C(t) : t \in \mathbb{R}\}$  in Banach space  $X$ ,  $f : J \times X \times X \rightarrow X$ ,  $k : J \times J \times X \rightarrow X$ ,  $g : J \times X \rightarrow X$ ,  $q, p : B \rightarrow X$  are appropriate continuous functions, and  $x_0, y_0$  are given elements of  $X$ .

The work on nonlocal initial value problem (IVP for short) was initiated by Byszewski. In [2] Byszewski, using the method of semigroups and the Banach fixed point theorem proved the existence and uniqueness of mild, strong and classical solution of first order IVP. For the importance of nonlocal conditions in different fields, the interesting reader is referred to [1, 3, 7] and the references cited therein. The theorem proved in this paper generalizes some results obtained by Eduardo Hernandez M. in [4, 5]. We are motivated by the work of Eduardo Hernandez M. in [4] and influenced by the work of Byszewski [2].

The paper is organized as follows: In Section 2, we present the preliminaries and hypotheses. Section 3 deals with the main result. Finally, in Section 4, we give an example of 'wave' equation to illustrate the application of our theorem.

## 2 Preliminaries and Hypotheses

In many cases it is advantageous to treat second abstract differential equations directly rather than to convert first order systems. A useful technique for the study of abstract second order equations is the theory of strongly continuous cosine family. We refer the reader to [8, 9] for the necessary concepts about cosine functions. Next, we only mention a few results and notations needed to establish our results. A one parameter family  $\{C(t) : t \in \mathbb{R}\}$  of bounded linear operators mapping the Banach space  $X$  into itself is called a strongly continuous cosine family if and only if

$$(a) \quad C(0) = I \quad (I \text{ is the identity operator});$$

$$(b) \quad C(t)x \text{ is strongly continuous in } t \text{ on } \mathbb{R} \text{ for each fixed } x \in X;$$

$$(c) \quad C(t+s) + C(t-s) = 2C(t)C(s) \text{ for all } t, s \in \mathbb{R}.$$

If  $\{C(t) : t \in \mathbb{R}\}$  is a strongly continuous cosine family in  $X$ , then  $\{S(t) : t \in \mathbb{R}\}$ , associated to the given strongly continuous cosine family, is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in \mathbb{R}. \quad (4)$$

For a closed operator  $G : D(G) \subset X \rightarrow X$  we denote by  $[D(G)]$  the space  $D(G)$  endowed with the graph norm  $\|\cdot\|_G$ . The infinitesimal generator  $A : X \rightarrow X$  of a cosine family  $\{C(t) : t \in \mathbb{R}\}$  is defined by

$$Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}, \quad x \in D(A),$$

where  $D(A) = \{x \in X : C(\cdot)x \in C^2(\mathbb{R}, X)\}$ . Moreover,  $M \geq 1$  and  $N$  are positive constants such that  $\|C(t)\| \leq M$  and  $\|S(t)\| \leq N$  for every  $t \in J$ .

If  $x(\cdot)$  is a solution of (1)–(3),  $g$  is  $D(A)$ -valued and  $Ag \in B$  then

$$\begin{aligned} x(t) &= C(t)[x_0 + q(x)] + S(t)[y_0 + p(x) + g(0, x_0 + q(x))] \\ &\quad - \int_0^t g(s, x(s))ds - \int_0^t AS(t-s) \int_0^s g(\tau, x_\tau)d\tau ds \\ &\quad + \int_0^t S(t-s)f\left(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau\right)ds, \quad t \in J. \end{aligned} \tag{5}$$

The expression (5) and the relation  $A \int_s^t S(\theta)x = C(t)x - C(s)x$  are the motivation of the following definition.

**Definition 2.1** *A function  $x \in B$  is a mild solution of the abstract nonlocal Cauchy problem (1)–(3) if condition (2) is verified and satisfy*

$$\begin{aligned} x(t) &= C(t)[x_0 + q(x)] + S(t)[y_0 + p(x) + g(0, x_0 + q(x))] \\ &\quad - \int_0^t C(t-s)g(s, x(s))ds + \int_0^t S(t-s) \\ &\quad \times f\left(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau\right)ds, \quad t \in J. \end{aligned} \tag{6}$$

We list the following hypotheses for our convenience.

(H<sub>1</sub>)  $X$  is a Banach space with norm  $\|\cdot\|$  and  $x_0, y_0 \in X$ .

(H<sub>2</sub>)  $t \in J$  and  $B_r = \{z : \|z\| \leq r\} \subset X$ .

(H<sub>3</sub>)  $f : J \times X \times X \rightarrow X$  satisfies the following conditions:

- (a) The function  $f(t, \cdot, \cdot) : X \times X \rightarrow X$  is continuous a.e.  $t \in J$ ,
- (b) The function  $f(\cdot, x, y) : J \rightarrow X$  is strongly measurable for each  $(x, y) \in X \times X$ ,
- (c) There exists a positive constant  $L_f$  such that

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq L_f(\|x_1 - x_2\| + \|y_1 - y_2\|),$$

for  $t \in J, x_i, y_i \in X$ .

(H<sub>4</sub>) The function  $g : J \times X \rightarrow X$  satisfies the following conditions:

- (d) The function  $g(t, \cdot) : X \rightarrow X$  is continuous a.e.  $t \in J$ ,
- (e) The function  $g(\cdot, x) : J \rightarrow X$  is strongly measurable for each  $x \in X$ ,
- (f) There exists a positive constant  $L_g$  such that

$$\|g(t, x_1) - g(t, x_2)\| \leq L_g \|x_1 - x_2\|, \quad \text{for } t \in J, x_i \in X,$$

- (g) There exist positive constants  $c_1$  and  $c_2$  such that

$$\|g(t, x)\| \leq c_1 \|x\| + c_2, \quad \text{for } t \in J, x \in X.$$

(H<sub>5</sub>)  $k : J \times J \times X \rightarrow X$  is continuous in  $t, s$  on  $J$  and there exists a constant  $K > 0$  such that

$$\|k(t, s, x_1) - k(t, s, x_2)\| \leq K \|x_1 - x_2\|, \quad \text{for } 0 \leq s \leq t \in J, x_i \in X.$$

(H<sub>6</sub>)  $q, p : B \rightarrow X$  are continuous and there exist constants  $L_q, L_p > 0$  such that

$$\begin{aligned} \|q(x_1) - q(x_2)\| &\leq L_q \|x_1 - x_2\|_b \quad \text{and} \\ \|p(x_1) - p(x_2)\| &\leq L_p \|x_1 - x_2\|_b, \quad \text{for } x_1, x_2 \in C(J, B_r). \end{aligned}$$

(H<sub>7</sub>)  $A$  is the infinitesimal generator of strongly continuous cosine family  $\{C(t) : t \in \mathbb{R}\}$  and  $S(t)$ , the sine function associated with  $C(t)$ , which is defined in (4).

(H<sub>8</sub>)  $L_1 = \max_{t \in J} \|f(t, 0, 0)\|$ ,  $K_1 = \max_{t \in J} \|k(t, s, 0)\|$ ,  $Q = \max_{x \in C(J, B_r)} \|q(x)\|$ ,  $P = \max_{x \in C(J, B_r)} \|p(x)\|$ .

(H<sub>9</sub>) The constants  $\|x_0\|, \|y_0\|, c_1, c_2, r, b, L_f, L_g, K, L_q, L_p, M, N, L_1, K_1, Q$ , and  $P$  satisfy the following inequalities:

$$\begin{aligned} M \left[ \|x_0\| + Q \right] + N \left[ \|y_0\| + P + c_1 (\|x_0\| + Q) + c_2 \right] + M(c_1 r + c_2) b \\ + N \left[ L_f r b + L_f K r b^2 + L_f K_1 b^2 + L_1 b \right] \leq r, \end{aligned}$$

and

$$\left[ M(L_q + L_g b) + N(L_p + L_g L_q + L_f b + L_f K b^2) \right] < 1.$$

Now, we establish our result of existence.

### 3 Existence of mild solution

**Theorem 3.1** *Suppose that the hypotheses  $[H_1] - [H_9]$  hold, then the problem (1)–(3) has a unique mild solution on  $J$ .*

Proof: Let  $Z = C(J, B_r)$  and define an operator  $F : Z \rightarrow Z$  by

$$\begin{aligned} (Fz)(t) &= C(t)[x_0 + q(z)] + S(t)[y_0 + p(z) + g(0, x_0 + q(z))] \\ &\quad - \int_0^t C(t-s)g(s, z(s))ds + \int_0^t S(t-s) \\ &\quad \times f\left(s, z(s), \int_0^s k(s, \tau, z(\tau))d\tau\right)ds, \quad t \in J. \end{aligned} \quad (7)$$

We shall first show that  $F$  maps  $Z$  into itself. To this end, from the definition of the operator  $F$  in (7) and our hypotheses, we obtain

$$\begin{aligned} &\|(Fz)(t)\| \\ &\leq \|C(t)[x_0 + q(z)]\| + \|S(t)[y_0 + p(z) + g(0, x_0 + q(z))]\| \\ &\quad + \int_0^t \|C(t-s)g(s, z(s))\|ds + \int_0^t \|S(t-s)f\left(s, z(s), \int_0^s k(s, \tau, z(\tau))d\tau\right)\|ds \\ &\leq M[\|x_0\| + Q] + N[\|y_0\| + P + \|g(0, x_0 + q(z))\|] + M \int_0^t \|g(s, z(s))\|ds \\ &\quad + N \int_0^t \left[\|f\left(s, z(s), \int_0^s k(s, \tau, z(\tau))d\tau\right) - f(s, 0, 0) + f(s, 0, 0)\right]ds \\ &\leq M[\|x_0\| + Q] + N[\|y_0\| + P + c_1(\|x_0\| + \|q(z)\|) + c_2] + M \int_0^t (c_1r + c_2)ds \\ &\quad + N \int_0^t \left[L_f\left(r + \int_0^s \|k(s, \tau, z(\tau)) - k(s, \tau, 0)\|d\tau + \int_0^s \|k(s, \tau, 0)\|d\tau\right) + L_1\right]ds \\ &\leq M[\|x_0\| + Q] + N[\|y_0\| + P + c_1(\|x_0\| + Q) + c_2] + M(c_1r + c_2)b \\ &\quad + N \int_0^t \left[L_f\left(r + \int_0^s Kr d\tau + K_1b\right) + L_1\right]ds \\ &\leq M[\|x_0\| + Q] + N[\|y_0\| + P + c_1(\|x_0\| + Q) + c_2] + M(c_1r + c_2)b \\ &\quad + N[L_frb + L_fKrb^2 + L_fK_1b^2 + L_1b] \\ &\leq r, \end{aligned} \quad (8)$$

for  $z \in Z$  and  $t \in J$ . Hence  $F(Z) \subset Z$ . Therefore, the equation (8) shows that the operator  $F$  maps  $Z$  into itself.

Now, we shall show that  $F$  is a contraction on  $Z$ . Then for this every  $z_1, z_2 \in Z$  and  $t \in J$ , we obtain

$$\|(Fz_1)(t) - (Fz_2)(t)\|$$

$$\begin{aligned}
 &\leq \|C(t)\| \|q(z_1) - q(z_2)\| \\
 &\quad + \|S(t)\| \left[ \|p(z_1) - p(z_2)\| + \|g(0, x_0 + q(z_1)) - g(0, x_0 + q(z_2))\| \right] \\
 &\quad + \int_0^t \|C(t-s)\| \left[ \|g(s, z_1(s)) - g(s, z_2(s))\| \right] ds \\
 &\quad + \int_0^t \|S(t-s)\| \left[ \|f(s, z_1(s), \int_0^s k(s, \tau, z_1(\tau))d\tau) \right. \\
 &\quad \left. - f(s, z_2(s), \int_0^s k(s, \tau, z_2(\tau))d\tau) \right] ds \\
 &\leq ML_q \|z_1 - z_2\|_b + N \left[ L_p \|z_1 - z_2\|_b + L_g \|q(z_1) - q(z_2)\| \right] \\
 &\quad + ML_g \|z_1 - z_2\|_b b + N \int_0^t L_f \left[ \|z_1 - z_2\|_b + K \|z_1 - z_2\|_b b \right] ds \\
 &\leq \left[ M(L_q + L_g b) + N(L_p + L_g L_q + L_f b + L_f K b^2) \right] \|z_1 - z_2\|_b. \tag{9}
 \end{aligned}$$

If we take  $l = \left[ M(L_q + L_g b) + N(L_p + L_g L_q + L_f b + L_f K b^2) \right]$ , then

$$\|Fz_1 - Fz_2\|_b \leq l \|z_1 - z_2\|_b$$

with  $0 < l < 1$ . This shows that the the operator  $F$  is a contraction on the complete metric space  $Z$ . By the Banach fixed point theorem, the function  $F$  has a unique fixed point in the space  $Z$  and this point is the mild solution of problem (1)–(3) on  $J$ .

### 4 Application

In this section, we give an example of the partial differential equation to illustrate of the application of our main theorem.

$$\frac{\partial}{\partial t} \left[ \frac{\partial w(t, u)}{\partial t} + g_1(t, w(t, u)) \right] = \frac{\partial^2 w(t, u)}{\partial u^2} + \mu \left( t, w(t, u), \int_0^t a(t, s, w(s, u)) ds \right),$$

$$t \in J, \quad u \in I = [0, \pi], \tag{10}$$

$$w(t, 0) = w(t, \pi) = 0, \quad t \in J, \tag{11}$$

$$w(0, u) = x_0(u) + \sum_{i=1}^n \alpha_i w(t_i, u), \quad u \in I, \tag{12}$$

$$\frac{\partial w(t, u)}{\partial t} \Big|_{t=0} = y_0(u) + \sum_{i=1}^k \beta_i w(s_i, u), \quad u \in I, \tag{13}$$

where  $\mu : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $g_1 : J \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $a : J \times J \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous and  $0 < t_i, s_i < b$ ,  $\alpha_i, \beta_i \in \mathbb{R}$  are prefixed numbers. Let us take  $X = L^2([0, \pi])$ .

We define the operator  $A : D(A) \subset X \rightarrow X$  by  $Aw = w_{uu}$ , where  $D(A) = \{w(\cdot) \in X : w(0) = w(\pi) = 0\}$ . It is well known that  $A$  is the generator of strongly continuous cosine function  $\{C(t) : t \in \mathbb{R}\}$  on  $X$ . Furthermore,  $A$  has discrete spectrum, the eigenvalues are  $-n^2, n \in \mathbb{N}$ , with corresponding normalized characteristics vectors  $w_n(u) := \sqrt{\frac{2}{\pi}} \sin(nu), n = 1, 2, 3, \dots$ , and the following conditions hold :

- (i)  $\{w_n : n \in \mathbb{N}\}$  is an orthonormal basis of  $X$ .
- (ii) If  $w \in D(A)$  then  $Aw = -\sum_{n=1}^{\infty} n^2 \langle w, w_n \rangle w_n$ .
- (iii) For  $w \in X, C(t)w = \sum_{n=1}^{\infty} \cos(nt) \langle w, w_n \rangle w_n$ . Moreover, from these expression, it follows that  $S(t)w = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle w, w_n \rangle w_n$ , that  $S(t)$  is compact for every  $t > 0$  and that  $\|C(t)\| \leq 1$  and  $\|S(t)\| \leq 1$  for every  $t \in J$ .
- (iv) If  $H$  denotes the group of translations on  $X$  defined by  $H(t)x(u) = \tilde{x}(u + t)$ , where  $\tilde{x}$  is the extension of  $x$  with period  $2\pi$ , then  $C(t) = \frac{1}{2}(H(t) + H(-t))$ . Hence it follows, see [6], that  $A = G^2$ , where  $G$  is the infinitesimal generator of the group  $H$  and that  $E = \{x \in L^1(0, \pi) : x(0) = x(\pi) = 0\}$ .

Define the functions  $f : J \times X \times X \rightarrow X, g : J \times X \rightarrow X, a : J \times J \times X \rightarrow X$ , and  $q, p : C(J, X) \rightarrow X$  as follows  $f(t, x, y)u = \mu(t, x(u), y(u)), g(t, x)u = g_1(t, x(u)), k(t, s, x)u = a(t, s, x(u)), q(x)u = \sum_{i=1}^n \alpha_i x(t_i, u), p(x)u = \sum_{i=1}^k \beta_i x(s_i, u), u \in C(I : X)$ , for  $0 < t_i, s_i < b$  and  $0 \leq u \leq \pi$ . Assume these functions satisfy the requirement of hypotheses. From the above choices of the functions and generator  $A$ , the equations (10)–(13) can be formulated as an abstract nonlinear second order integrodifferential equations (1)–(3) in Banach space  $X$ . Since all hypotheses of the Theorem 3.1 are satisfied, therefore, the Theorem 3.1 can be applied to guarantee the solution of the nonlinear partial integrodifferential equation (10)–(13).

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