Applied Mathematical Sciences, Vol. 3, 2009, no. 43, 2119 - 2132

On Linear Systems Containing Strict Inequalities in Reflexive Banach Spaces

E. Naraghirad

Department of Mathematics of Yasouj University
Yasouj, Postal code: 75914, Iran
eskandarrad@gmail.com

Abstract

In this paper, we consider linear systems of an arbitrary number of both weak and strict inequalities in a reflexive Banach space X. The number of inequalities and equalities in these systems is arbitrary (possibly infinite). For such kind of systems a consistency theorem is provided and those strict inequalities (weak inequalities, equalities) which are satisfied for every solution of a given system are characterized. The main result of the present paper is to characterize the consistent linear systems containing strict inequalities without any assumption. The results presented in this paper generalize the corresponding results of Miguel A. Goberna, Margarita M. L. Rodriguez, Analyzing linear systems containing strict inequalities via evenly convex hulls, European Journal of Operational Research 169 (2006) 1079-1095.

Mathematics Subject Classification: 90C05; 90C25; 90C34

Keywords: Convex Programming, Linear Programming, Evenly convex set, Evenly convex hull, Semi-infinite Programming

1 Introduction

Linear inequality systems in the finite-dimensional space $\mathbb{R}^n$ containing an arbitrary number of either weak or strict inequalities have been investigated in [4]. Such kind of systems can be written as

$$\pi = \{a_t'x \geq b_t,\ t \in W;\ a_t'x > b_t,\ t \in S\}, \tag{1. 1}$$

where $W \cup S \neq \emptyset,\ W \cap S = \emptyset,\ a_t \in \mathbb{R}^n$ and $b_t \in \mathbb{R}$ for all $t \in V := W \cup S$. The solution set of π is denoted by F .
There exists an extensive literature on ordinary linear inequality systems ($S =$

$\emptyset$, $\mid W \mid < \infty$) as far as they are closely related to linear programming theory and methods. Concerning linear semi-infinite systems ($S = \emptyset$, $W \neq \emptyset$ arbitrary), whose analysis provides the theoretical foundations for linear semi-infinite programming (LISP), different conditions for $F \neq \emptyset$ (existence theorems) and many results characterizing the geometrical properties of F in terms of coefficients of σ are well-known (see [5, Part II] and references therein).

Evenly convex sets were introduced by Fenchel in 1952 to extend the polarity theory. A set $C \subset \mathbb{R}^n$ is said to be *evenly convex* (in [3]) if it is the intersection of a family of open half-spaces. Since this family can be empty, $\mathbb{R}^n$ and $\emptyset$ are evenly convex sets.
On the other hand, since any closed half-space is the intersection of infinitely many open half-spaces, C is evenly convex if and only if C is the solution set of a certain linear inequality system such as (1.1). In particular, any closed convex set is evenly convex. Given an evenly convex set C containing 0_n (0_n represents the zero vector in $\mathbb{R}^n$), its modified (negative) polar was defined by Fenchel as $C^0 = \{y \in \mathbb{R}^n : x'y < 1 \quad \forall x \in C\}$, proving that $C^{00} = C$. On the other hand, given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\pm\infty\}$, f is said to be evenly quasi-convex if $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is evenly convex for all $\alpha \in \mathbb{R}$. New characterizations of these functions have been given in [1]. This class of functions has applications in quasi-convex programming (duality and conjugacy, [8,9,12,13]) and mathematical economy (consumer theory, [10]).

Let X be a locally convex Hausdorff topological vector space. Define by X^* the dual space of X and by X^{**} the second dual of X. Every element x of X gives rise to a continuous linear functional $\hat{x}$ on X^* by the formula

$$\hat{x}(f) = f(x) \qquad \text{for all } f \in X^*.$$

The mapping $Q_X : X \longrightarrow X^{**}$ defined by

$$Q_X(x) = \hat{x} \qquad (x \in X),$$

is called the natural embedding of X into its second dual X^{**}. The mapping Q_X is onto if and only if X is a reflexive Banach space (see [11, Definition 1.11.6]). Let X be a Banach space and $X^* \times \mathbb{R}$ be the product space, with the norm defined by

$$\| (f,b) \| = \| f \| + \mid b \mid \qquad ((f,b) \in X^* \times \mathbb{R}).$$

Similarly, $X^{**} \times \mathbb{R}$ is a Banach space under the norm

$$\| (\Phi,c) \| = \| \Phi \| + \mid c \mid \qquad ((\Phi,c) \in X^{**} \times \mathbb{R}).$$

By $< (x,c),(f,b) >_1 := f(x) + bc$ we denote the dual coupling for any $x \in X$, $f \in X^*$ and $b, c \in \mathbb{R}$ and $< (f,b),(\Phi,d) >_2 := \Phi(f) + bd$ for any $\Phi \in X^{**}$, $f \in X^*$ and $b, d \in \mathbb{R}$.

In this paper, we consider linear inequality systems in a reflexive Banach space X of the form

$$\sigma = \{f_t(x) > b_t, t \in S; f_t(x) \geq b_t, t \in W; f_t(x) = b_t, t \in E\} \tag{1. 2}$$

with $S \neq \emptyset$, S, W and E pairwise disjoint sets, $b_t \in \mathbb{R}$ and $f_t \in X^*$ for all $t \in T := S \cup W \cup E$(a possibly infinite set). We denote $V = S \cup W$. The main purpose of the paper is to characterize those systems σ for which there exist solutions (i.e., the class of consistent systems). The secondary purpose consists of characterizing those (weak and strict) inequalities and equalities which are satisfied by every solution of σ. In other words, this problem consists of characterizing the consequence inequalities or equalities of a given consistent system.

In this section, we introduce the necessary notations. Given a non-empty set E in X, we denote by conv$E$, cone$E$, aff$E$ and span$E$ the convex hull of E, the convex cone generated by E, the affine hull of E and the linear subspace of X spanned by E, respectively. Moreover, we define cone$\emptyset$=span$\emptyset$=$\{0\}$. If $E \neq \emptyset$ is a convex cone, $E^0 = \{f \in X^* : f(x) \geq 0, \ \forall \ x \in E\}$ denotes the positive polar cone of E. Moreover, from the topological side, we denote by cl$E$, int$E$, rbd$E$ and rint$E$ the closure, the interior, the relative boundary and the relative interior of E, respectively. Define by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_+^{(T)}$, the real numbers, the non-negative real numbers and the space of generalized finite sequences, whose elements are the functions $\lambda : T \longrightarrow \mathbb{R}_+$ such that $\lambda_t \neq 0$ only on a finite subset of T, respectively.

The structure of this paper is as follows. In section 2, we give some preliminary results on evenly convex sets and prove that the product of evenly convex sets is evenly convex too. In section 3, we study linear systems in a reflexive Banach space X containing an arbitrary number of either weak or strict inequalities. In section 4, we give the relationships between a linear system σ as in (1.2) and a linear relation $f(x) \geq b$ ($f(x) > b$ or $f(x) = b$) in a reflexive Banach space X.

2 Preliminaries

Let X be a locally convex Hausdorff topological vector space. In this section, we consider linear inequality systems in X of the form

$$\pi = \{f_t(x) \geq b_t, \ t \in W; \ f_t(x) > b_t, \ t \in S\} \tag{2. 1}$$

with $W \cup S \neq \emptyset$, $W \cap S = \emptyset$, $f_t \in X^*$ and $b_t \in \mathbb{R}$ for all $t \in V := W \cup S$.

A set $C \subset X$ is said to be *evenly convex* (in [3]) if it is the intersection of a family of open half-spaces. Since this family can be empty, X and $\emptyset$ are evenly convex sets.

On the other hand, since any closed half-space is the intersection of infinitely many open half-spaces, C is evenly convex if and only if C is the solution set of a certain linear inequality system such as (2.1). In particular, any closed convex set is evenly convex.

If conv$A \subset X$, conv$A \neq X$, the intersection of all the open half-spaces containing A is the minimal evenly convex set which contains A, *i.e.*, it is eco$A$. Alternatively, if conv$A = X$ (*i.e.*, if there does not exist a half-space containing A), then eco$A = X$. Obviously, A is evenly convex if and only if eco$A = A$. This happens, for instance, if A is either a closed or a relatively open convex set. Consequently, if A is a compact (open) set, then conv$A$ is a compact (open) convex set and eco$A$=conv$A$. This is the case, in particular, if $\mid A \mid< \infty$. From the definition of evenly convex set, given $\bar{x} \in X$, $\bar{x} \notin$ eco$A$ if and only if there exists $f \in X^*$ such that $f(x-\bar{x}) > 0$, for all $x \in X$. In particular,

$$0 \notin \text{eco}A \quad \text{if and only if} \quad \{f(x) > 0, x \in A\} \text{ is consistent.} \tag{2. 2}$$

It can be easily seen that, for any $A \subset X$,

$$\text{ecoconv}A = \text{eco}A = \text{conveco}A. \tag{2. 3}$$

Proposition 2.1. *Let X and Y be locally convex Hausdorff topological vector spaces and $C_1 \subset X$, $C_2 \subset Y$ be two evenly convex sets. Then the cartesian product $C_1 \times C_2$ is also evenly convex.*

Proof: If

$$C_1 = \{x \in X : f_u(x) > b_u,\ u \in U\}$$

and

$$C_2 = \{y \in Y : g_v(y) > d_v,\ v \in V\},$$

then

$$C_1{\times}C_2 = \{(x,y) \in X{\times}Y : (f_u,0)(x,y) > b_u,\ u \in U;\ (0,g_v)(x,y) > d_v,\ v \in V\},$$

where for any $f \in X^*$ and $g \in Y^*$, $(f,g)(x,y) = f(x)+g(y)$, $(x,y) \in X \times Y$. Hence $C_1 \times C_2$ is evenly convex, which completes the proof.■

Proposition 2.2. *Let X and Y be locally convex Hausdorff topological vector spaces. Given two sets $A \subset X$ and $B \subset Y$,*

$$\text{eco}(A \times B) = (\text{eco}A) \times (\text{eco}B).$$

Proof: According to Proposition 2.1, the product of evenly convex sets is evenly convex. This implies that

$$\text{eco}(A \times B) \subset (\text{eco}A) \times (\text{eco}B).$$

For the opposite inclusion, we assume $(\bar{x}, \bar{y}) \notin \text{eco}(A \times B)$. Let $(0, 0) \neq (f, g) \in X^* \times Y^*$ and $\alpha \in \mathbb{R}$ such that $f(x) + g(y) > \alpha$ for all $(x, y) \in A \times B$ and

$$f(\bar{x}) + g(\bar{y}) \leq \alpha \tag{2. 4}$$

(or even $f(\bar{x}) + g(\bar{y}) = \alpha$). We have either $f \neq 0$ or $g \neq 0$(maybe both).
If $f \neq 0$ and $g = 0$, then $f(x) > \alpha$ for all $x \in A$ and $f(\bar{x}) \leq \alpha$, so that $\bar{x} \notin \text{eco}A$ and $(\bar{x}, \bar{y}) \notin \text{eco}A \times \text{eco}B$.
We get the same conclusion if $f = 0$ and $g \neq 0$.
Then, we can assume $f \neq 0$ and $g \neq 0$. Let $y \in B$ and $D := \{x \in X : f(x) > \alpha - g(y)\}$. Then D is an open half-space containing A, so that, if $\bar{x} \in \text{eco}A$, we have $f(\bar{x}) > \alpha - g(y)$. This, together with (2.4), implies that $g(y) > g(\bar{y})$ for all $y \in B$, so that $\bar{y} \notin$ B and $(\bar{x}, \bar{y}) \notin \text{eco}A \times \text{eco}B$.
So $(\bar{x}, \bar{y}) \notin \text{eco}A \times \text{eco}B$ in either case and we obtain $(\text{eco}A) \times (\text{eco}B) \subset \text{eco}(A \times B)$, which completes the proof.■

Lemma 2.1. *Let X be locally convex Hausdorff topological vector space and A, B, C be nonempty sets in X. Then the following statements hold:*
(i) $\text{conv}A+\text{cone}B=\text{conv}(A + \mathbb{R}_+B)$.
(ii) $\text{conv}A+\text{cone}B+\text{span}C=\text{conv}(A + \mathbb{R}_+B + \mathbb{R}C)$.

Proof: The proof is similar to that of [6, Lemma 2.1] and we omit it.■

3 Existence of solutions

In this section, we study linear systems in a reflexive Banach space X of the form (2.1). In the following we give the main result of the paper which characterizes the consistent linear systems containing strict inequalities without any assumption.

Proposition 3.1. *Let X be a reflexive Banach space. Let π be the system in (2.1).*
(i) *If π is consistent, then*

$$(0, 1) \notin \text{clcone}\{(f_t, b_t) : t \in T\}. \tag{3. 1}$$

Moreover, if $S \neq \emptyset$, then the following statement also holds:

$$(0,0) \notin \text{conv}\{(f_t, b_t) : t \in S\} + \text{cone}\{(f_t, b_t) : t \in W\}. \qquad (3.\ 2)$$

(ii) *Each of the following conditions guarantees the consistency of* π*:*
(ii.a) $S = \emptyset$ *and (3.1) holds.*
(ii.b) $S \neq \emptyset$*, (3.1) and (3.2) hold and the set in (3.2) is closed.*

Proof: (i) Assume on the contrary that

$$(0,1) \in \text{clcone}\{(f_t, b_t) : t \in T\}.$$

Then there exists a sequence $\{(g_n, \alpha_n)\}_{n=1}^{\infty} \subset X^* \times \mathbb{R}$ such that

$$\lim_{n \longrightarrow \infty} \| (g_n, \alpha_n) - (0,1) \| = 0$$

and

$$(g_n, \alpha_n) = \sum_{t \in T} \lambda_t^n (f_t, b_t), \qquad \lambda_t^n \in \mathbb{R}_+^{(T)}, \qquad n = 1, 2, 3, \ldots.$$

Since π is consistent, we can take $x_0 \in F$. Then, we have

$$g_n(x_0) - \alpha_n = < (x_0, -1), (g_n, \alpha_n) >_1 = \sum_{t \in T} \lambda_t^n < (x_0, -1), (f_t, b_t) >_1$$

$$= \sum_{t \in T} \lambda_t^n (f_t(x_0) - b_t) \geq 0, \qquad n = 1, 2, 3, \ldots \qquad (3.\ 3)$$

and, taking limits in (3.3), we get the contradiction $-1 \geq 0$. Hence, (3.1) holds.

Now assume that $S \neq \emptyset$. If

$$(0,0) \in \text{conv}\{(f_t, b_t) : t \in S\} + \text{cone}\{(f_t, b_t) : t \in W\},$$

then there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that $\sum_{t \in S} \lambda_t = 1$ and

$$(0,0) = \sum_{t \in S} \lambda_t (f_t, b_t) + \sum_{t \in W} \lambda_t (f_t, b_t).$$

Let x_0 be an arbitrary solution of π. Then, we get

$$0 = < (x_0, -1), (0,0) >_1 = \sum_{t \in S} \lambda_t (f_t(x_0) - b_t) + \sum_{t \in W} \lambda_t (f_t(x_0) - b_t) > 0.$$

Hence, (3.2) holds.
(ii.a) Assume that $S = \emptyset$ and (3.1) holds. By a well-known Corollary of the Hahn-Banach theorem, there exist $\Phi \in X^{**}$ and $\alpha,\ \gamma \in \mathbb{R}$, such that $<(0,1),(\Phi,\alpha)>_2 = \gamma$ and $<(f,b),(\Phi,\alpha)>_2 > \gamma$ for all $(f,b) \in \text{cone}\{(f_t,b_t) : t \in W\}$. The last condition implies that

$$<(f,b),(\Phi,\alpha)>_2 \geq 0 > \gamma \quad \text{ for all } (f,b) \in \text{cone}\{(f_t,b_t) : t \in W\},$$

in particular, $<(f_t,b_t),(\Phi,\alpha)>_2 \geq 0$ for all $t \in W$. This means that

$$\Phi(f_t) + \alpha b_t \geq 0 \quad \text{ for all } t \in W. \tag{3. 4}$$

Since X is reflexive, in view of [10, Definition 1.11.6], there exists $x_1 \in X$ such that $\Phi = \hat{x_1}$. This, together with (3.4), implies that

$$f_t(x_1) + \alpha b_t \geq 0 \qquad \text{ for all } t \in W. \tag{3. 5}$$

Let $x = \mid \alpha \mid^{-1} x_1$. Since $\alpha = \gamma < 0$ and $x \in X$, from (3.5), we conclude that $f_t(x) \geq b_t$ for all $t \in W$, so that π is consistent.
(ii.b) Now, let us assume that $S \neq \emptyset$, (3.1) and (3.2) hold and the set

$$A := \text{conv}\{(f_t,b_t) : t \in S\} + \text{cone}\{(f_t,b_t) : t \in W\}$$

is closed.
Since $(0,0) \notin A$, by a well-known Corollary of the Hahn-Banach theorem, there exist $\Phi \in X^{**}$ and $\alpha \in \mathbb{R}$, such that $<(f,b),(\Phi,\alpha)>_2 > 0$ for all $(f,b) \in A$. Since $(f_s,b_s) \in A$ for all $s \in S$, $\Phi(f_t) + \alpha b_t > 0$ for all $t \in S$. Since X is reflexive, in view of [11, Definition 1.11.6], there exists $x_1 \in X$ such that $\Phi = \hat{x_1}$. This implies that $f_s(x_1) + \alpha b_s = \Phi(f_s) + \alpha b_s > 0$ for all $s \in S$. Since $(f_s,b_s) + \beta(f_t,b_t) \in A$ for all $s \in S$, $t \in W$ and $\beta > 0$, we get

$$f_t(x_1) + \alpha b_t > -\frac{f_s(x_1) + \alpha b_s}{\beta}.$$

Hence, $f_t(x_1) + \alpha b_t \geq 0$ for all $t \in W$.
Let $\bar{x}$ be a solution of $\bar{\pi}$ (system satisfying (ii.a) and, so, consistent) and consider the following point of X:

$$x := \begin{cases} \frac{x_1}{|\alpha|} & \text{if} \quad \alpha < 0 \\ x_1 + \bar{x} & \text{if} \quad \alpha = 0 \\ 2\bar{x} + \frac{x_1}{\alpha} & \text{if} \quad \alpha > 0. \end{cases}$$

It is easy to show that $x \in F$, so that π is consistent, which completes the proof.■

The following two results are immediate consequences of Proposition 3.1, [2, Theorem 1] and [15, Theorem 2].

Proposition 3.2. *Let X be a reflexive Banach space. A system* $\{f_t(x) \geq b_t, t \in W; f_t(x) = b_t, t \in E\}$ is consistent if and only if

$$(0,1) \notin \text{clcone}\{(f_t, b_t), t \in W; \pm(f_t, b_t), t \in E\}.$$

Proposition 3.3. *Let X be a reflexive Banach space. A weak inequality $f(x) \geq b$ is a consequence of a consistent system $\{f_t(x) \geq b_t, t \in W; f_t(x) = b_t, t \in E\}$ if and only if $(f, b) \in \text{cl}K$, where*

$$K = \text{cone}\{(f_t, b_t), t \in W; \pm(f_t, b_t), t \in E; (0, -1)\}.$$

Consequently, a strict inequality $f(x) > b$ is a consequence of a consistent system $\{f_t(x) \geq b_t, t \in W; f_t(x) = b_t, t \in E\}$ if and only if

$$(f, b) \in \text{cl}K \qquad \text{and} \qquad (0, 1) \in \text{cl}[K + \text{span}\{(f, b)\}].$$

For the sake of brevity in the proofs, given a subset of indices $I \subset T$, we denote by C_I the set $\{(f_t, b_t), t \in I\}$.

Lemma 3.1. *Let X be a reflexive Banach space. The following conditions are equivalent to each other:*
(i) $\{f_t(x) > b_t, t \in S\}$*, with $S \neq \emptyset$, is consistent;*
(ii) $(0, 0) \notin \text{eco}[\{(f_t, b_t), t \in S; (0, -1)\}]$*;*
(iii) $(0, 0) \notin \text{eco}[\{(f_t, b_t), t \in S\}]$ and $(0, 1) \notin \text{clcone}\{(f_t, b_t), t \in S\}$

Proof: (i) $\Longrightarrow$ (ii) Let $x_1 \in X$ be a solution of $\{f_t(x) > b_t, t \in S\}$. Then $(x_1, -1) \in X \times \mathbb{R}$ is a solution of the homogeneous system

$$\{< (x, \alpha), (f_t, b_t) >_1 > 0, t \in S; < (x, \alpha), (0, -1) >_1 > 0\},$$

whose consistency is equivalent to condition (ii) by (2.2).
(ii) $\Longrightarrow$ (iii) Since $\text{eco}C_S \subset \text{eco}[C_S \cup \{(0, -1)\}]$, condition (ii) entails $(0, 0) \notin \text{eco}C_S$.
On the other hand, condition (ii) and a well-known Corollary of the Hahn-Banach theorem imply that there exist $\Phi \in X^{**}$ and $\alpha \in \mathbb{R}$ such that $< (f, b), (\Phi, \alpha) >_2 > 0$ for all $(f, b) \in C_S$ and $< (0, -1), (\Phi, \alpha) >_2 > 0$. Let $B := \{(f, b) \in X^* \times \mathbb{R} :< (f, b), (\Phi, \alpha) >_2 \geq 0\}$. Then B is an homogeneous closed half-space containing C_S such that $(0, 1) \notin B$, and, by an argument similar to that of [14, Corollary 11.7.2], $\text{clcone}C_S \subset B$. Consequently, $(0, 1) \notin \text{clcone}C_S$

and (iii) holds.
(iii) $\Longrightarrow$ (i) Now assume that $(0,0) \notin \text{eco}C_S$ and $(0,1) \notin \text{clcone}C_S$.
It follows, from the first condition, that there exists a vector $(\Phi, \alpha) \in X^{**} \times \mathbb{R}$ such that $< (f,b), (\Phi,\alpha) >_2 > 0$ for all $(f,b) \in C_S$. Then

$$\Phi(f_t) + \alpha b_t > 0 \qquad \text{for all } t \in S. \tag{3. 6}$$

In view of Proposition 3.2, second condition is equivalent to the consistency of the system $\{f_t(x) \geq b_t, t \in S\}$. Let $\bar{x}$ be a solution of such system.
It can be easily realized, from (3.6), that $x \in X$, defined as

$$x := \begin{cases} \frac{y}{|\alpha|} & \text{if} \quad \alpha < 0 \\ y + \bar{x} & \text{if} \quad \alpha = 0 \\ 2\bar{x} + \frac{y}{\alpha} & \text{if} \quad \alpha > 0 \end{cases}$$

is a solution of $\{f_t(x) > b_t, t \in S\}$, which completes the proof.■

Theorem 3.1. *Let X be a reflexive Banach space. The following conditions are equivalent to each other:*
(i) $\sigma = \{f_t(x) > b_t, t \in S; f_t(x) \geq b_t, t \in W; f_t(x) = b_t, t \in E\}$, *with $S \neq \emptyset$ is consistent;*
(ii) $(0,0) \notin \text{eco}[\{(f_t,b_t), t \in S\} + \mathbb{R}_+\{(f_t,b_t), t \in W\} + \mathbb{R}\{(f_t,b_t), t \in E\}; (0,-1)]$;
(iii) $(0,0) \notin \text{eco}[\{(f_t,b_t), t \in S\} + \mathbb{R}_+\{(f_t,b_t), t \in W\} + \mathbb{R}\{(f_t,b_t), t \in E\}]$
and $(0,1) \notin \text{clcone}\{(f_t,b_t), t \in V; \pm(f_t,b_t), t \in E\}$.

Proof: Without loss of generality, we can assume $W \neq \emptyset \neq E$ (otherwise the proof is simpler). The proof is a straightforward consequence of Lemma 3.1 due to the equivalence between the consistency of σ and the consistency of the system

$$\sigma_1 = \{(f_s + \alpha f_w + \beta f_e)(x) > b_s + \alpha b_w + \beta b_e,\ (s,w,e,\alpha,\beta) \in S \times W \times E \times \mathbb{R}_+ \times \mathbb{R}\}.$$

Since any solution of σ is trivially a solution of σ_1, we consider a solution of σ_1, say $\bar{x}$.
Given $s \in S$, and taking arbitrary $w \in W$ and $e \in E$, we have, for the index $(s,w,e,0,0) \in S \times W \times E \times \mathbb{R}_+ \times \mathbb{R}$, the strict inequality

$$f_s(\bar{x}) > b_s. \tag{3. 7}$$

Similarly, given $w \in W$, taking arbitrary $s \in S$, $e \in E$ and $r > 0$, we have, for the index $(s,w,e,r,0) \in S \times W \times E \times \mathbb{R}_+ \times \mathbb{R}$, the inequality

$$(f_s + r f_w)(\bar{x}) > b_s + r b_w. \tag{3. 8}$$

Multiplying by r^{-1} both sides of (3.8) and taking limits as $r \longrightarrow \infty$, we get

$$f_w(\bar{x}) > b_w. \tag{3. 9}$$

Finally, given $e \in E$, for any $s \in S$, $w \in W$ and $r \in \mathbb{R} \setminus \{0\}$, we have, for the index $(s, w, e, 0, r) \in S \times W \times E \times \mathbb{R}_+ \times \mathbb{R}$, the inequality

$$(f_s + rf_e)(\bar{x}) > b_s + rb_e. \tag{3. 10}$$

Multiplying by $| \ r \ |^{-1}$ both sides of (3.10) and taking limits as $r \longrightarrow \infty$, we get now

$$f_e(\bar{x}) = b_e. \tag{3. 11}$$

Therefore, from (3.7), (3.9) and (3.11), we conclude that $\bar{x}$ is a solution of σ, which completes the proof.■

Obviously, conditions (ii) and (iii) are equivalent to assert the consistency of associated systems in $X^* \times \mathbb{R}$ (in the same vein, the alternative theorems establish the equivalence between the consistency of a given system and the consistency of an associated one). Now we shall establish some straightforward consequences of Theorem 3.1.

Corollary 3.1. *Let X be a reflexive Banach space. Let σ be as in Theorem 3.1, with $E = \emptyset$. Then the following statements hold:*
(i) *If σ is consistent, then*

$$(0, 1) \notin \text{clcone}\{(f_t, b_t) : t \in T\} \tag{3. 12}$$

and

$$(0, 0) \notin \text{conv}\{(f_t, b_t) : t \in S\} + \text{cone}\{(f_t, b_t) : t \in W\}. \tag{3. 13}$$

(ii) *If (3.12) and (3.13) hold and the set in (3.2) is closed, then σ is consistent.*

Proof: (i) According to Lemma 2.1, part (i), we have

$$\text{conv}C_S + \text{cone}C_W = \text{conv}[C_S + \mathbb{R}_+ C_W] \subset \text{eco}[C_S + \mathbb{R}_+ C_W]$$

and (3.12) and (3.13) hold by straightforward application of Theorem 3.1.
(ii) We can reformulate (3.13) as

$$(0, 1) \notin \text{conv}[C_S + \mathbb{R}_+ C_W]. \tag{3. 14}$$

Since we are assuming the closedness of the set in (3.14), it is equal to the set $\text{eco}[C_S + \mathbb{R}_+ C_W]$. Then Theorem 3.1 applies again, which completes the proof.■

The dubious case in Corollary is illustrated in [6, Example 3.1].

Corollary 3.2. *Let X be a reflexive Banach space. Let σ be as in Theorem 3.1, with $E=\emptyset$. Assume that $\{f_t(x)\geq b_t, t\in W\}$ is consistent and that* $\operatorname{cone}\{(f_t,b_t), t\in T\}$ *and* $\operatorname{conv}\{(f_t,b_t), t\in S\}+\operatorname{cone}\{(f_t,b_t), t\in W\}$ *are closed sets. Then one of the following alternatives holds:*
(i) *σ is consistent.*
(ii) *There exists $\lambda\in\mathbb{R}_+^{(T)}$ such that at least one of the numbers λ_t, $t\in S$, is nonzero, and*

$$\sum_{t\in T}\lambda_t f_t=(0,0) \qquad \text{and} \qquad \sum_{t\in T}\lambda_t b_t\geq 0. \tag{3. 15}$$

Proof: We shall prove the equivalence between (ii) and the negation of (i). According to Corollary 3.1, (i) fails if and only if either

$$(0,0)\in \operatorname{cone}C_T \tag{3. 16}$$

or

$$(0,0)\in \operatorname{conv}C_S+\operatorname{cone}C_W. \tag{3. 17}$$

If (3.16) holds, we can write

$$(0,1)=\sum_{t\in T}\lambda_t(f_t,b_t), \qquad \lambda\in\mathbb{R}_+^{(T)},$$

so that (3.15) holds. Moreover, there exists $t\in S$ such that $\lambda_t>0$ (otherwise $(0,1)\in\operatorname{cone}C_W$, and this is impossible due to the consistency of $\{f_t(x)\geq b_t, t\in W\}$). Alternatively, if (3.17) holds, there exists $\lambda\in\mathbb{R}_+^{(T)}$ such that

$$(0,0)=\sum_{t\in T}\lambda_t(f_t,b_t) \qquad \text{and} \qquad \sum_{t\in S}\lambda_t=1$$

and (3.15) holds again with $\lambda_t>0$ for at least one index $t\in S$. Consequently, (ii) holds in both cases.
Now we assume that (ii) holds, i.e., there exist $\lambda\in\mathbb{R}_+^{(T)}$ and $\alpha\in\mathbb{R}_+$ such that $\sum_{t\in S}\lambda_t>0$ and

$$(0,\alpha)=\sum_{t\in T}\lambda_t(f_t,b_t). \tag{3. 18}$$

If $\alpha > 0$, dividing both members of (3.18) by α, we get

$$(0,1) = \sum_{t \in T} (\alpha^{-1}\lambda_t)(f_t, b_t) \in \text{cone}C_T$$

and σ is inconsistent.
Alternatively, if $\alpha = 0$, dividing both members of (3.18) by $\mu := \sum_{t \in S} \lambda_t$, we get

$$(0,0) = \sum_{t \in S} (\mu^{-1}\lambda_t)(f_t, b_t) + \sum_{t \in W} (\mu^{-1}\lambda_t)(f_t, b_t) \in \text{conv}C_S + \text{cone}C_W,$$

so that σ is inconsistent by Corollary 3.1.
Hence, (i) fails in both cases, which completes the proof.■

The next result generalizes the Extended Motzkin's Alternative Theorem (see [5, p.68]).

Corollary 3.3. *Let X be a reflexive Banach space. An homogeneous system*

$$\sigma = \{f_t(x) > 0, t \in S; f_t(x) \geq 0, t \in W; f_t(x) = 0, t \in E\}$$

is consistent if and only if

$$(0,0) \notin \text{eco}[\{f_t, t \in S\} + \mathbb{R}_+\{f_t, t \in W\} + \mathbb{R}\{f_t, t \in E\}]. \tag{3. 19}$$

Proof: In view of Theorem 3.1 and Proposition 2.2, σ is consistent if and only if

$$(0,0) \notin \text{eco}[\{(f_t, 0), t \in S\} + \mathbb{R}_+\{(f_t, 0), t \in W\} + \mathbb{R}\{(f_t, 0), t \in E\}]$$

$$= \text{eco}[\{f_t, t \in S\} + \mathbb{R}_+\{f_t, t \in W\} + \mathbb{R}\{f_t, t \in E\}] \times \{0\}$$

if and only if (3.19) holds.■

4 Consequence relations

A linear relation $f(x) \geq b(f(x) > b$ or $f(x) = b)$ is a consequence of a system σ if $f(\bar{x}) \geq b(f(\bar{x}) > b$ or $f(\bar{x}) = b)$ holds for every $\bar{x} \in X$ solution of σ. If σ is inconsistent, then any linear inequality (or equality) is a consequence of σ. So we assume throughout this section that

$$\sigma = \{f_t(x) > b_t, t \in S; f_t(x) \geq b_t, t \in W; f_t(x) = b_t, t \in E\}$$

is consistent. We denote $\bar{\sigma} = \{f_t(x) \geq b_t, t \in V; f_t(x) = b_t, t \in E\}$.

Proposition 4.1. *Let X be a reflexive Banach space. A weak inequality $f(x) \geq b$ is a consequence of σ if and only if*

$$(f, b) \notin \text{clcone}\{(f_t, b_t), t \in V; \pm(f_t, b_t), t \in E; (0, -1)\}. \qquad (4.\ 1)$$

Proof: If $f(x) \geq b$ is a consequence of σ, then it is also a consequence of $\bar{\sigma}$. Indeed, let $\bar{x}$ be an arbitrary solution of $\bar{\sigma}$ and take an arbitrary x_1 solution of σ. Then, for every $\lambda \in (0, 1)$, we get a solution of σ of the form $x(\lambda) := (1-\lambda)\bar{x} + \lambda x_1$. Since $f(x) \geq b$ is a consequence of σ, we get $f(x(\lambda)) \geq b$ and taking limits as $\lambda \longrightarrow 0^+$ we obtain $f(\bar{x}) \geq b$. Since $f(x) \geq b$ is a consequence of a consistent system $\bar{\sigma} = \{f_t(x) \geq b_t, t \in V; f_t(x) = b_t, t \in E\}$, in view of Proposition 3.3 we obtain (4.1), which completes the proof.■

Corollary 4.1. *Let X be a reflexive Banach space. An equation $f(x) = b$ is a consequence of σ if and only if*

$$\pm(f, b) \in \text{clcone}\{(f_t, b_t), t \in V; \pm(f_t, b_t), t \in E; (0, -1)\}.$$

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Received: January, 2009