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Stability of RTS of Efficient DMUs in DEA with Fuzzy u_0 under Fuzzy Data

Adel Hatami-Marbini

Department of Industrial Engineering-Productivity and System Management,
Tehran-South Branch, Islamic Azad University, Tehran, Iran
adel_hatami@yahoo.com

Saber Saati

Department of Mathematics, Tehran-North Branch, Islamic Azad University, Tehran, Iran

Abstract. An important property of production functions is the concept return to scale (RTS) as found in the literature. There are two common variations RTS in data envelopment analysis (DEA) used, constant return to scale (CRS) and variation return to scale (VRS). The envelopment surface in BCC model is VRS and this is the result of the presence of the convexity constraint in the dual model and, equivalently, the presence of new separate variable, usually called u_0, is introduced in the primal model which makes it possible to determine whether operations were led in the areas of constant, increasing and decreasing. In conventional BCC model make an assumption that input-output data and u_0 are exact. While accurate measurement in many real applications due to either non-availability of sophisticated measurement tools or qualitative nature of the phenomena may not be possible, consequently, this information can be represented as fuzzy numbers or linguistic terms.

In this paper the RTS of efficient decision making units (DMUs) are investigated in BCC model where fuzziness is considered in both inputs and outputs and u_0 variable. The proposed models provide the stability of fuzzy u_0 as interval region, consequently analyst can easily examine the sensitivity of RTS of efficient DMUs. Using α -cut, an alternative approach is suggested to solve the obtained model. To illustrate the proposed method, a numerical example is solved.

Mathematics Subject Classification: 90C70

Keywords: Data Envelopment Analysis; Fuzzy mathematical programming; BBC model; Fuzzy u_0; Stability of RTS

1. Introduction

The evaluation of production activities has long been recognized to be a problem of considerable complexity. However, a variety of tools are available to measure technical efficiency that one of the most common tool also is data envelopment analysis (DEA)), (see [1] and [2]).

DEA is a mathematic programming method for evaluating the relative efficiency of a set of homogenous decision-making units (DMUs) on the basis of multiple inputs and outputs. This technique was first proposed by Charnes, Cooper and Rhodes (CCR model) [2]. During this period of model development, the economic notion of returns-to-scale (RTS) was widely studied within the framework of DEA. This, in turn, further extended the applicability of DEA. Banker, Charnes and Cooper (BCC) [3] report that a new free BCC dual variable (u_0) estimates RTS by allowing variable returns-to-scale (VRS) for the CCR model, i.e. the sign of u_0 determines the RTS. Banker and Thrall [4] also generalized the BCC RTS method by exploring all alternate optima in the BCC dual model, i.e. RTS in their extended technique is measured by intervals for u_0.

The original DEA models such as CCR and BCC assume that all data are known exactly without any variation. In real decision-making and evaluation problems, ordinal preference information and/or fuzzy data are often encountered. When all or parts of input and output data are fuzzy data, several approaches have been proposed to deal with the fuzzy data in the framework of DEA [5-12], especially some scholars proposed several fuzzy BCC models [13-15]. One of the assumptions of all existent fuzzy BCC models is based on crisp u_0 whereas this assumption in fuzzy environment cannot be correct. In other words, the sign of crisp u_0 cannot be credible for identifying RTS.

The main purpose of this paper is to develop a new fuzzy BCC model with fuzzy u_0. We transform the fuzzy DEA model to interval programming by applying an alternative α*-level* approach so that a family of conventional crisp DEA models could be utilized. In this method, instead of comparing the equality (or inequality) of two intervals, a variable were defined in the interval, not only satisfies the set of constraints, but also optimizes the efficiency value. Thereby, we determine RTS status of DMUs which are efficient. The potential of our method enables us to examine thoroughly RTS status of efficient units.

The rest of the paper is organized as follows: DEA and fuzzy DEA models are introduced in Section 2. Section 3 presents' models to obtaining upper and lower of RTS indicator for efficient DMU while variable of u_0 is considered in the fuzzy form. This is followed by presentation an empirical study to illustrate proposed models. Finally, Section 5 concludes the paper.

2. DEA models based on exact data

The most frequently used DEA models are CCR and BCC models [2-3]. Suppose that there are *n* DMUs. Each DMU_j, $j=1,\ldots,n$ produces *s* different outputs y_{rj} $(r=1,\ldots,s)$, using *m* different inputs x_{ij} $(i=1,\ldots,m)$. All inputs and outputs are assumed to be nonnegative, but at least one input and one output are positive.

The primal linear programming problem for the (input-based) CCR model can be written as:

$$\begin{aligned} \max \quad & \theta_p = \sum_{r=1}^{s} u_r y_{rp} \\ s.t. \quad & \sum_{i=1}^{m} v_i x_{ip} = 1, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad \forall j, \\ & \quad u_r, v_i \ge 0, \quad \forall r, i. \end{aligned} \tag{1}$$

where u_r $(r=1,\ldots,s)$ and v_i $(i=1,\ldots,m)$ are the output and input weights assigned the *r*th output and *i*th input, respectively and, $p \in \{1,\ldots,n\}$ is index of the DMU under evaluation.

A constant return to scale (CRS) is assumed in (1). Banker et al. [3] modify (1) to suit for cases of variable returns to scale. The VRS DEA model is formulated as the following linear programming model:

$$\begin{aligned} \max \quad & w_p = \sum_{r=1}^{s} u_r y_{rp} + u_0 \\ s.t. \quad & \sum_{i=1}^{m} v_i x_{ip} = 1, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \le 0, \quad \forall j, \\ & \quad u_r, v_i \ge 0, \quad \forall r, i. \end{aligned} \tag{2}$$

When u_0 is set to zero, the assumption of constant returns to scale is imposed, and (2) becomes (1). In this paper, the focus will be on the BCC model because it determines the RTS of DMUs.

3. Fuzzy RTS

Recently, fuzzy set theory has been proposed as a way to quantify imprecise and vague data in DEA models. Fuzzy DEA models take the form of fuzzy linear programming models. The VRS DEA model with fuzzy coefficients and u_0 can be written as:

$$
\begin{aligned}
\max \quad & w_p = \sum_{r=1}^{s} u_r \tilde{y}_{rp} + \tilde{u}_0 \\
s.t. \quad & \sum_{i=1}^{m} v_i \tilde{x}_{ip} = \tilde{1}, \\
& \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} + \tilde{u}_0 \le \tilde{0}, \quad \forall j, \\
& \quad u_r, v_i \ge 0, \quad \forall r, i.
\end{aligned} \tag{3}
$$

where, " ~ " indicates the fuzziness.

Among the several types of fuzzy numbers, triangular fuzzy numbers are of more common in application. In the sequel, let $\tilde{x}_{ij} = (x_{ij}^{M}, x_{ij}^{L}, x_{ij}^{R})$, $\tilde{y}_{rj} = (y_{rj}^{M}, y_{rj}^{L}, y_{rj}^{R})$ and $\tilde{u}_0 = (u_0^{M}, u_0^{L}, u_0^{R})$. Therefore, (3) can be written as follows:

$$
\begin{aligned}
\max \quad & w_p = \sum_{r=1}^{s} u_r (y_{rp}^{M}, y_{rp}^{L}, y_{rp}^{R}) + (u_0^{M}, u_0^{L}, u_0^{R}) \\
s.t. \quad & \sum_{i=1}^{m} v_i (x_{ip}^{M}, x_{ip}^{L}, x_{ip}^{R}) = (1^{M}, 1^{L}, 1^{R}), \\
& \sum_{r=1}^{s} u_r (y_{rj}^{M}, y_{rj}^{L}, y_{rj}^{R}) - \sum_{i=1}^{m} v_i (x_{ij}^{M}, x_{ij}^{L}, x_{ij}^{R}) + (u_0^{M}, u_0^{L}, u_0^{R}) \le (0^{M}, 0^{L}, 0^{R}), \ \forall j, \\
& \quad u_r, v_i \ge 0, \quad \forall r, i.
\end{aligned} \tag{4}
$$

Model (4) is a possibilistic linear programming. Several approaches have been proposed to deal with it [6-15]. Here we apply the proposed approach in [8]. Therefore, the following model would be achieved:

$$
\begin{aligned}
\max \quad & w_p = \sum_{r=1}^{s} u_r \hat{y}_{rp} + \hat{u}_0 \\
s.t. \quad & \sum_{i=1}^{m} v_i \hat{x}_{rp} = \hat{1}, \\
& \sum_{r=1}^{s} u_r \hat{y}_{rj} - \sum_{i=1}^{m} v_i \hat{x}_{ij} + \hat{u}_0 \le \hat{0}, \quad \forall j, \\
& \alpha x_{ij}^{M} + (1-\alpha) x_{ij}^{L} \le \hat{x}_{ij} \le \alpha x_{ij}^{M} + (1-\alpha) x_{ij}^{R}, \quad \forall i, j, \\
& \alpha y_{rj}^{M} + (1-\alpha) y_{rj}^{L} \le \hat{y}_{rj} \le \alpha y_{rj}^{M} + (1-\alpha) y_{rj}^{R}, \quad \forall r, j, \\
& \alpha u_0^{M} + (1-\alpha) u_0^{L} \le \hat{u}_0 \le \alpha u_0^{M} + (1-\alpha) u_0^{R}, \\
& \alpha 1^{M} + (1-\alpha) 1^{L} \le \hat{1} \le \alpha 1^{M} + (1-\alpha) 1^{R}, \\
& \alpha 0^{M} + (1-\alpha) 0^{L} \le \hat{0} \le \alpha 0^{M} + (1-\alpha) 0^{R}, \\
& \quad \hat{y}_{rj}, \hat{x}_{ij}, \hat{1}, \hat{0} \ge 0, \quad \forall r, i, j, \\
& \quad u_r, v_i \ge 0, \quad \forall r, i.
\end{aligned} \tag{5}
$$

In (5), the objective value w_p can exceed one with change of right hand side (RHS) of the constrain $\sum_{i=1}^{m} v_i \hat{x}_{rp} = \hat{1}$. For normalization of the objective value w_p, RHS of the first constrain of (5) is assumed to be equal to 1. Therefore, RHS of the first constrain of (5) must be equal to 1. Similarly, we must consider $\hat{0}$ equal to 0. Model (5) is a nonlinear programming problem. In order to linearize this model, following substitutions are performed:

$\bar{x}_{ij} = v_i \hat{x}_{ij}$, $\bar{y}_{rj} = u_r \hat{y}_{rj} \qquad \forall r, i, j.$

Therefore, (5) is transformed to the following linear programming problem:

$$
\begin{aligned}
\max \quad & w_p = \sum_{r=1}^{s} \bar{y}_{rp} + \hat{u}_0 \\
s.t. \quad & \sum_{i=1}^{m} \bar{x}_{ip} = 1, \\
& \sum_{r=1}^{s} \bar{y}_{rj} - \sum_{i=1}^{m} \bar{x}_{ij} + \hat{u}_0 \le 0, \qquad \forall j, \\
& v_i \left(\alpha x_{ij}^{M} + (1-\alpha) x_{ij}^{L} \right) \le \bar{x}_{ij} \le v_i \left(\alpha x_{ij}^{M} + (1-\alpha) x_{ij}^{R} \right), \qquad \forall i, j, \\
& u_r \left(\alpha y_{rj}^{M} + (1-\alpha) y_{rj}^{L} \right) \le \bar{y}_{rj} \le u_r \left(\alpha y_{rj}^{M} + (1-\alpha) y_{rj}^{R} \right), \qquad \forall r, j, \\
& \alpha u_0^{M} + (1-\alpha) u_0^{L} \le \hat{u}_0 \le \alpha u_0^{M} + (1-\alpha) u_0^{R}, \\
& \quad \bar{y}_{rj}, \bar{x}_{ij} \ge 0, \qquad \forall r, i, j, \\
& \quad u_r, v_i \ge 0, \qquad \forall r, i.
\end{aligned}
\tag{6}
$$

which is a parametric linear programming problem, while $\alpha \in [0,1]$ is a parameter.

Under fuzzy data assumption, to investigate the RTS of efficient DMUs we used the results of suggested approach in [4]. It is argued that there may be production possibilities where there are more than one (in fact, infinite) supporting hyperplanes. Hence, we need to determine the lower and upper bounds of $\hat{u}_0$ on the slopes of all such supporting hyperplanes. Hence, to determine the upper bound of $\hat{u}_0$ the following model is considered:

$$
\begin{aligned}
\max \quad & u_0^{R} \\
s.t. \quad & \sum_{r=1}^{s} \bar{y}_{rp} + \hat{u}_0 = 1, \\
& \sum_{i=1}^{m} \bar{x}_{ip} = 1, \\
& \sum_{r=1}^{s} \bar{y}_{rj} - \sum_{i=1}^{m} \bar{x}_{ij} + \hat{u}_0 \le 0, \qquad \forall j,
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
& v_i\left(\alpha x_{ij}^{M}+(1-\alpha)x_{ij}^{L}\right)\le \bar{x}_{ij}\le v_i\left(\alpha x_{ij}^{M}+(1-\alpha)x_{ij}^{R}\right), && \forall i,j,\\
& u_r\left(\alpha y_{rj}^{M}+(1-\alpha)y_{rj}^{L}\right)\le \bar{y}_{rj}\le u_r\left(\alpha y_{rj}^{M}+(1-\alpha)y_{rj}^{R}\right), && \forall r,j,\\
& \alpha u_0^{M}+(1-\alpha)u_0^{L}\le \hat{u}_0\le \alpha u_0^{M}+(1-\alpha)u_0^{R},\\
& \quad \bar{y}_{rj},\bar{x}_{ij}\ge 0, \quad \forall r,i,j,\\
& \quad u_r,v_i\ge 0, \quad \forall r,i.
\end{aligned}
$$

Due to the constrain $\sum_{r=1}^{s}\bar{y}_{rp}+\hat{u}_0=1$, (7) is feasible for only efficient DMUs.

By solving (7) for a given $\alpha\in[0,1]$, an upper bound for $\hat{u}_0$ namely u_0^R would determined for the DMU under consideration. A similar minimization model may be solved to determine the lower bound on $\hat{u}_0$, namely u_0^L. Hence, an interval $[u_0^L,u_0^R]$ is achieved for $\tilde{u}_0$ to determine RTS of efficient DMUs. It can be easily shown that $u_0^L\le u_0^R$ and the optimal value of these models can be unbounded.

To classify the RTS of an efficient DMU we proceed as follows:

1. If $u_0^R<0$, then DMU_p is decreasing returns to scale (DRS) in each given α-cut,
2. If $u_0^L>0$, then DMU_p is increasing returns to scale (IRS) in each given α-cut,
3. If $u_0^R.u_0^L\le 0$, then DMU_p is constant returns to scale (CRS) in each given α-cut.

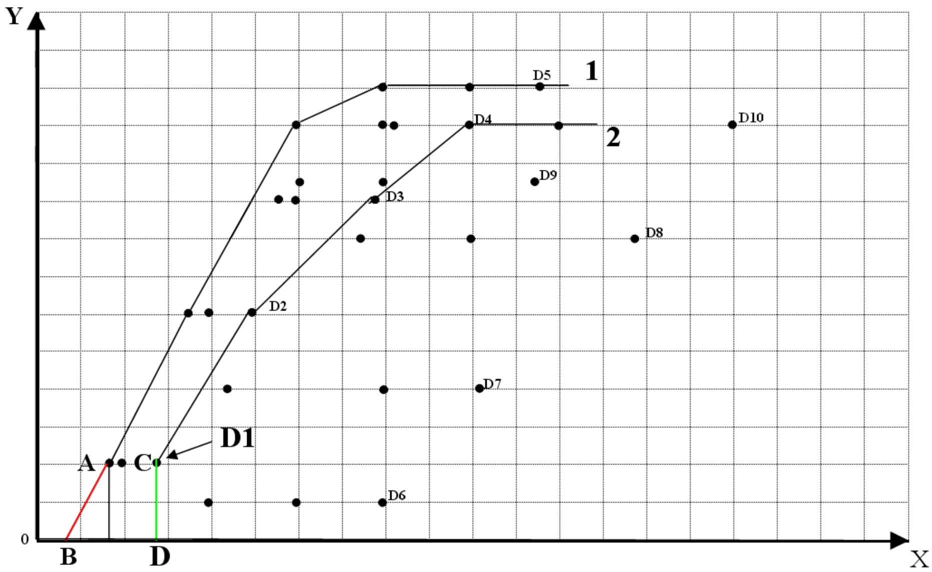

Table 1. Input and output data of 10 DMUs

DMU	Input	Output
	Triangular fuzzy number	Crisp
D1	(1,0.9,1.3)	(1,1,1)
D2	(2,1.78,2.5)	(3,3,3)
D3	(3,2.8,3.9)	(4.5, 4.5,4.5)
D4	(4,3.1,5)	(5.5,5.5,5.5)
D5	(5,4,5.75)	(6,6,6)
D6	(3,1.9,3.99)	(0.5,0.5,0.5)
D7	(4,2.2,5.1)	(2,2,2)
D8	(5,3.7,6.9)	(4,4,4)
D9	(4,3.1,5.7)	(4.75, 4.75, 4.75)
D10	(6,4.1,8)	(5.5, 5.5, 5.5)

4. Illustrative example

In this section, to illustrate the suggested approach, a simple numerical example is considered. 10 DMUs are considered with one triangular fuzzy input and one crisp output. These data are listed in Table 1.

In Figure 1, these DMUs are shown. The two solid lines (1) and (2) represent production frontier. The lower and upper bounds of $\tilde{u}_0$ are calculated with suggested models. From Table 2, we see that D1 is IRS for all $\alpha \in [0,1]$ and D2 and D3 are CRS under any α-level. Also, it must be noted that DMUs may be efficient for only some given α. In Figure 1, (AB) and (CD) lines show upper and lower hyperplanes of production possibilities for D1, respectively. It is obvious that the result of obtained from proposed models is the same result of shown in Figure 1 for D1. Moreover, we can consider the variation of the RTS situation for a given efficient DMU by changing α.

5. Conclusions

Fuzzy data envelopment analysis (FDEA) is a tool to assess the performance of a set of activities or organizations under uncertainty environment. One of the assumptions of all existent fuzzy BCC models is based on crisp u_0 whereas this assumption in fuzzy environment cannot be correct. In other words, the sign of crisp u_0 cannot be credible for identifying RTS.

In this paper we propose a method to deal with RTS of efficient DMUs in fuzzy BCC that is considered u_0 variable in the fuzzy form. Whereas efficient DMUs always lie on the efficient frontier; it may be possible to define their RTS situation in each α*-level*. These models determine the stability for fuzzy u_0 as an interval $[u_0^L, u_0^R]$ for efficient DMUs and it is also used for sensitivity analysis of RTS. Therefore, the variation of the RTS situation for a given efficient DMU can be easily verified by diverse α s.

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DMU_j	$\alpha = 0$			$\alpha = 0.2$		
	u_0 values		RTS status	u_0 values		RTS status
	u_0^L	u_0^R		u_0^L	u_0^R	
D1	0.11	1	IRS	0.25	1.25	IRS
D2	-1.17	0.6	CRS	-1.2	0.65	CRS
D3	-2.16	0.31	CRS	-2.39	0.35	CRS
D4	-8.4	0.29	CRS	-8.48	0.3	CRS
D5	-∞	0.19	CRS	-∞	0.18	CRS
D9	-2.25	0.26	CRS	-2.11	0.27	CRS
D10	-3.43	0.14	CRS	-2.19	-0.03	DRS

$\alpha = 0.5$			$\alpha = 0.7$			$\alpha = 1$		
u_0 values		RTS status	u_0 values		RTS status	u_0 values		RTS status
u_0^L	u_0^R		u_0^L	u_0^R		u_0^L	u_0^R	
0.63	2	IRS	1.31	3.33	IRS	0.5	1	IRS
-1.3	0.83	CRS	-1.27	1.15	CRS	0	0.25	CRS
-2.97	0.45	CRS	-3.61	0.64	CRS	-0.5	0	CRS
-9.31	0.35	CRS	-1	0.23	CRS	-1.75	-0.38	DRS
-∞	0	CRS	-∞	-0.72	DRS	-∞	-1.4	DRS
-1.39	0.01	CRS	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–

Table 2. RTS evaluation for efficient DMUs