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The Solitary Wave Solutions of a GIBE

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Abstract

Under the condition $\gamma > 0$,the solitary waves are studied of the GIBE by using qualitative theory of ordinary differential equations and numerical simulation method. The parameter conditions that the solitary waves appear are found, and their solutions are obtained. The planar graphs of the traveling wave equation are simulated by use mathematical software *Maple*. The numerical simulation and qualitative results are identical.

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1 Introduction

In recent years, exact soliton solutions to a large number of nonlinear partial differential equations have been found. It is the realization that such solutions possess special solutions in the form of pulses which retain their shapes and velocities after interaction between themselves. The existence of such stable solutions is due to a balance between dispersion and nonlinearity in the partial differential equations. For more details about solutions, consult texts such as [1,3,4]. An example of a soliton-producing equation is Boussinesq equation (BE)

$$u_t = u_x + (u^2)_x + u_{xxx} \tag{1}$$

which describes shallow water waves propagating in both directions [1]. In [5] Z. T. Fu studied a generalized Boussinesq equation (GBE) and obtained two exact soliton solutions. If the term $(u^2)_{xx}$ of Eq.(1) is replaced by $(u^3)_{xx}$, then this equation may be denoted the modified improved Boussinesq equation (IBE) [2].

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In this paper, we investigate the solitary waves of a nonlinear quartic equation with form

$$u_t = \alpha u_x + \beta(u^3)_x + \gamma u_{xxx}, \tag{2}$$

where $\gamma > 0, \alpha, \beta \in R$. Obviously, it is a family of generalized improved Boussinesq equation(GIBE). Proceeding to the ideas in [6,7,8], Firstly, the Eq.(2) is changed to a planar system, the properties of the singular points are studied, and the bifurcation phase portraits are drew. Finally, The parameter conditions that the solitary waves appear are found, and their solutions are obtained, the planar graphs of the traveling wave equation are simulated by use mathematical software *Maple*.

The rest of this paper is organized as follows. In Section 2, we firstly derive the traveling wave equation and the traveling wave system. Then we study the bifurcations of phase portrait of the traveling wave system. In Section 3, using the information of phase portrait, we make the numerical simulation for bounded integral curves of the traveling wave equation. In Section 4, we obtain the analytic expressions of the solitary wave solutions from the bifurcations of phase portrait and the bounded integral curves. Finally, a short conclusion is given in Section 5.

2 Bifurcation Phase Portraits

We substitute the traveling wave solution $u(x,t) = \phi(\xi)$ with $\xi = x - ct$ into (2) for constant wave speed c, and obtain the following ordinary differential equation

$$-c\phi' = \alpha\phi' + \beta(\phi^3)' + \gamma\phi'''. \tag{3}$$

Integrating once, we have

$$\gamma\phi'' = -(c+\alpha)\phi - \beta\phi^3 + g, \tag{4}$$

where g is integral constant. Letting $\phi'(\xi) = y$, we obtain a planar autonomous system

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{-(c+\alpha)\phi - \beta\phi^3 + g}{\gamma}. \tag{5}$$

Obviously, the (5) is a Hamiltonian system with Hamiltonian function

$$H(\phi, y) = \frac{1}{2}y^2 + \frac{1}{\gamma}(\frac{c+\alpha}{2}\phi^2 + \frac{\beta}{4}\phi^4 - g\phi) = h. \tag{6}$$

Let

$$f_0(\phi) = -(c+\alpha)\phi - \beta\phi^3, \tag{7}$$

and

$$\phi^*_\pm = \pm\sqrt{-\frac{c+\alpha}{3\beta}}, \quad for \quad \frac{c+\alpha}{3\beta} < 0. \tag{8}$$

It is easy to see that $\phi^*_\pm$ are extreme points of the $f_0(\phi)$ when $\frac{c+\alpha}{3\beta} < 0$. Let $g_\pm = f_0(\phi^*_\pm)$, then we have

$$g_\pm = \mp\frac{2(c+\alpha)}{3}\sqrt{-\frac{c+\alpha}{3\beta}}, \quad for \quad \frac{c+\alpha}{3\beta} < 0. \tag{9}$$

Let

$$f(\phi) = \frac{1}{\gamma}(f_0(\phi) + g), \tag{10}$$

and $(\phi_0, 0)$ be one of singular of the system (5). Then at the singular point $(\phi_0, 0)$ characteristic values of the system (5) are

$$\lambda_{1,2} = \pm\sqrt{\frac{1}{\gamma}f_0'(\phi_0)}. \tag{11}$$

Thus, we know that if $f_0'(\phi_0) > 0$, then the $(\phi_0, 0)$ is saddle point, if $f_0'(\phi_0) < 0$, then the $(\phi_0, 0)$ is center point, if $f_0'(\phi_0) = 0$ and $-(c+\alpha) \neq 0$, then the $(\phi_0, 0)$ is degenerate saddle point.

From $f_0'(\phi) = -3\beta\phi^2 - (c+\alpha)$, we have

1). $f_0'(\phi) > 0$, under one of the following four conditions. i). $\beta < 0$ and $c + \alpha \leq 0$, ii). $\beta < 0$, $c + \alpha > 0$ and $\phi < \phi^*_-$ or $\phi^*_+ < \phi$, iii). $\beta = 0$ and $c+\alpha < 0$, iv). $\beta > 0$, $c+\alpha < 0$ and $\phi^*_- < \phi < \phi^*_+$ (Fig.1. (a), (b), (c) and (e)).

2). $f_0'(\phi) < 0$, under one of the following four conditions. i). $\beta < 0$, $c + \alpha > 0$ and $\phi^*_- < \phi < \phi^*_+$, ii). $\beta = 0$, $c + p > 0$, iii). $\beta > 0$, $c + \alpha < 0$ and $\phi < \phi^*_-$ or $\phi^*_+ < \phi$, iv). $\beta > 0$, $c + \alpha \geq 0$. (Fig.1. (b), (d), (e) and (f)).

3). $f_0'(\phi) = 0$, when $\phi = \phi^*_\pm$ (Fig.1. (b) and (e)).

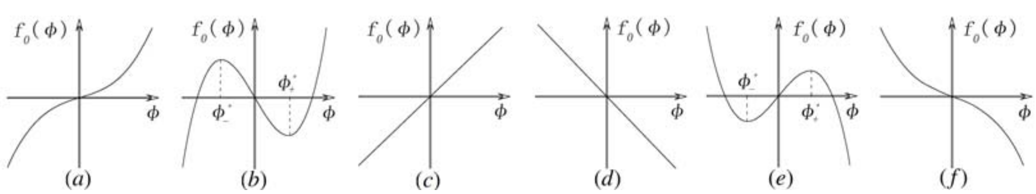

Fig.1. The sketch of $f_0(\phi)$ when $\alpha, \beta \in R$.
(a). $\beta < 0$ and $c + \alpha \leq 0$, (b). $\beta < 0$ and $c + \alpha > 0$, (c). $\beta = 0$ and $c + \alpha < 0$, (d). $\beta = 0$ and $c + \alpha > 0$, (e). $\beta > 0$ and $c + \alpha < 0$, (f). $\beta > 0$ and $c + \alpha \geq 0$.

According to the above analysis, we draw the bifurcation of phase portraits of the system (5) as Fig.2.

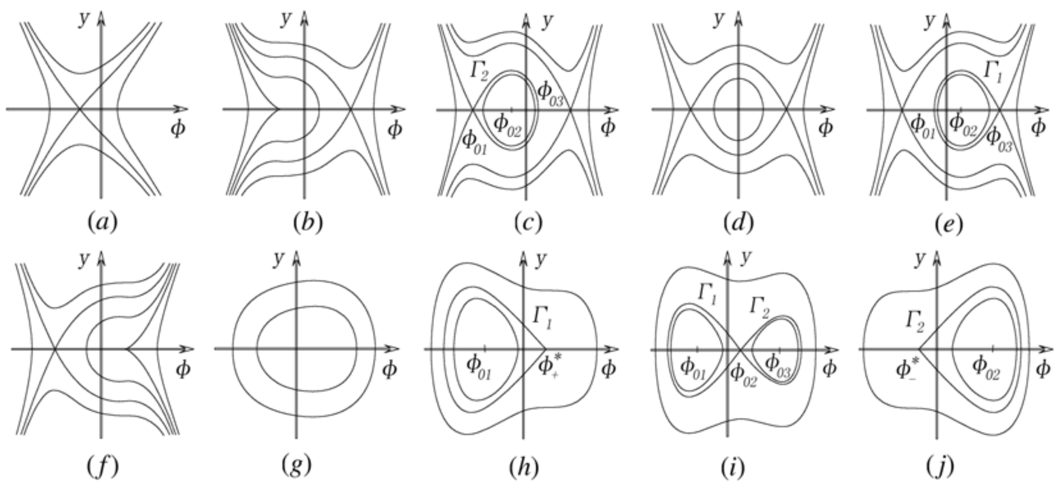

Fig.2. The bifurcation of phase portraits of the system (5).
(a). $(i).\beta < 0$ and $c + \alpha \leq 0$, $(ii).\beta < 0$, $c + \alpha > 0$ and $g > -g_-$ or $g < -g_+$, $(iii).\beta = 0$ and $c + \alpha < 0$. (b). $\beta < 0$, $c + \alpha > 0$ and $g = -g_-$. (c). $\beta < 0$, $c + \alpha > 0$ and $-g_- < g < 0$. (d). $\beta < 0$, $c + \alpha > 0$ and $g = 0$. (e). $\beta < 0$, $c + \alpha > 0$ and $0 < g < -g_+$. (f). $\beta < 0$, $c + \alpha > 0$ and $g = -g_+$. (g). $(i).\beta = 0$ and $c + p > 0$, $(ii).\beta > 0$, $c + \alpha < 0$ and $g > -g_-$ or $g < -g_+$, $(iii).\beta > 0$ and $c + \alpha \geq 0$. (h). $\beta > 0$, $c + \alpha < 0$ and $g = -g_+$. (i). $\beta > 0$, $c + \alpha < 0$ and $-g_+ < g < -g_-$. (j). $\beta > 0$, $c + \alpha < 0$ and $g = -g_-$.

3 Numerical Simulation

From the derivation in Sec.2 we see that the traveling waves of Eq.(2) correspond to the integral curves of Eq.(4), and the Homoclinic orbits of the systems (5) correspond to the bounded integral curves of Eq.(4) are the solitary waves of Eq.(2). Therefore, we can simulate the planar graphs of the solitary waves of Eq.(2) by using the information of the phase portrait of the systems (5). From Fig.2 it is seen that the systems (5) has Homoclinic orbits passing through the saddle point when one of the following five conditions is satisfied.

(H1). $\beta < 0$, $c + \alpha > 0$ and $-g_- < g < 0$.

(H2). $\beta < 0$, $c + \alpha > 0$ and $0 < g < -g_+$.

(H3). $\beta > 0$, $c + \alpha < 0$ and $g = -g_+$.

(H4). $\beta > 0$, $c + \alpha < 0$ and $g = -g_-$.

(H5). $\beta > 0$, $c + \alpha < 0$ and $-g_+ < g < -g_-$.

We simulate the bounded integral curves of Eq.(4) under the condition (H5) only by using mathematical software $Maple$, the other cases can use a similar argument. From Fig.2 (i), it is seen that the $(\phi_{02}, 0)$ is the saddle point of the systems (5). The systems (5) has two Homoclinic orbits Γ_1 and Γ_2 passing through the $(\phi_{02}, 0)$.

For example, choosing $\alpha = -5, c = 0.5, \beta = 1$ and $\gamma = 2$, we have $g_{\pm} \approx 3.674234613$. Taking $g = 2$, then $\phi_{21} \approx -2.459281633$, $\phi_{02} \approx -0.4670903792$ and $\phi_{23} \approx 3.393462392$ are roots of $H(\phi, 0) = H(\phi_{02}, 0)$.

Let $\phi(0) = -2.4, -2.452, -2.459$ and -2.459281633 respectively and $\phi'(0) = 0$. The bounded integral curves of Eq.(5) are simulated as (a), (b), (c) and (d) in Fig.3.

Let $\phi(0) = 3.37, 3.393, 3.3934$ and 3.393462392, respectively and $\phi'(0) = 0$. The bounded integral curves of Eq.(5) are simulated as (e), (f), (g) and (h) in Fig.3.

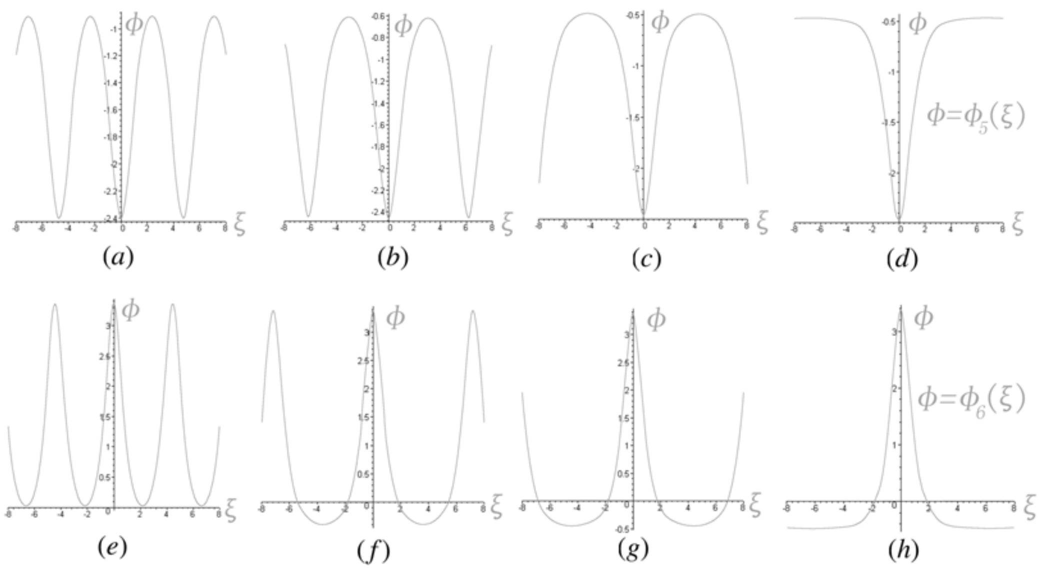

Fig.3. the bounded integral curves of Eq.(4) when
$\alpha = -5, \beta = 1, c = 0.5, \gamma = 2$ and $g = 2$.
(a). $\phi(0) = -2.4$. (b). $\phi(0) = -2.452$,. (c). $\phi(0) = -2.459$. (d). $\phi(0) = -2.459281633$. (e). $\phi(0) = 3.37$. (f). $\phi(0) = 3.393$. (g). $\phi(0) = 3.3934$. (h).$\phi(0) = 3.393462392$.

Remark 1. From Fig.3, we can see that the periodic smooth integral curves tend to a concave bell integral curve when $\phi_{21} < \phi < \phi_{02}$ and ϕ tends to ϕ_{21}, tend to a convex bell integral curve when $\phi_{02} < \phi < \phi_{23}$ and ϕ tends to ϕ_{23}.

4 Explicit solitary wave solutions

In this section, we derive the expressions of solitary wave solutions by using information obtained from above sections.

We let $(\phi_{01}, 0), (\phi_{02}, 0)$ and $(\phi_{03}, 0)$ are singular points of the system (5), and $(\phi_{0k}, 0)$ $(k = 1, 2, 3)$ is saddle point that homoclinic orbit passing through, then from (6) the homoclinic orbit has representation

$$H(\phi, y) = H(\phi_{0k}, 0). \tag{12}$$

Let

$$G_k(\phi) = H(\phi_{0k}, 0) - H(\phi, 0), \tag{13}$$

then from (12) and (13) we obtain the homoclinic orbit has representation

$$y = \pm\sqrt{2G_k(\phi)}. \tag{14}$$

From (13) we obtain

$$G_k'(\phi) = -H'(\phi, 0) = f(\phi). \tag{15}$$

Theorem 1. If one of hypotheses (H1 - H4) holds, then Eq.(2) has one solitary wave. If hypotheses H5 holds, then Eq.(2) has two solitary wave. And

1). Under condition H1. The solitary wave solution of Eq.(2) has expression $u_1(x,t) = \phi_1(\xi)$.

$$\phi_1(\xi) = \phi_{01} + \frac{2a_1}{b_1 + (\phi_{13} - \phi_{12})\cosh(e_1\xi)}. \tag{16}$$

Where $a_1 = (\phi_{12} - \phi_{01})(\phi_{13} - \phi_{01})$, $b_1 = \phi_{13} + \phi_{12} - 2\phi_{01}$, $e_1 = \sqrt{-\frac{\beta}{2\gamma}a_1}$, and the $(\phi_{01}, 0)$ is saddle point, ϕ_{01}, ϕ_{12} and ϕ_{13} are real roots of $G_1(\phi) = 0$, $\phi_{01} < \phi_{12} < \phi_{13}$.

2). Under condition H2. The solitary wave solution of Eq.(2) has expression $u_2(x,t) = \phi_2(\xi)$.

$$\phi_2(\xi) = \phi_{03} - \frac{2a_2}{b_2 + (\phi_{32} - \phi_{31})\cosh(e_2\xi)}. \tag{17}$$

Where $a_2 = (\phi_{03} - \phi_{31})(\phi_{03} - \phi_{32})$, $b_2 = -\phi_{31} - \phi_{32} + 2\phi_{03}$, $e_2 = \sqrt{-\frac{\beta}{2\gamma}a_2}$, and the $(\phi_{03}, 0)$ is saddle point, ϕ_{31}, ϕ_{32} and ϕ_{03} are real roots of $G_3(\phi) = 0$, $\phi_{31} < \phi_{32} < \phi_{03}$.

3). Under condition H3. The solitary wave solution of Eq.(2) has expression $u_3(x,t) = \phi_3(\xi)$.

$$\phi_3(\xi) = \phi_+^* - \frac{8\gamma(\phi_+^* - \phi_{21})}{8\gamma + \beta(\phi_+^* - \phi_{21})^2\xi^2}. \tag{18}$$

Where the $(\phi_+^*, 0)$ is degenerate saddle point, ϕ_{21} and ϕ_+^* are real roots of $G_2(\phi) = 0$ and $\phi_{21} < \phi_+^*$.

4). Under condition H4. The solitary wave solution of Eq.(2) has expression $u_4(x,t) = \phi_4(\xi)$.

$$\phi_4(\xi) = \phi_-^* + \frac{8\gamma(\phi_{23} - \phi_-^*)}{8\gamma + \beta(\phi_{23} - \phi_-^*)^2\xi^2}. \tag{19}$$

Where the $(\phi_-^*, 0)$ is degenerate saddle point, ϕ_-^* and ϕ_{23} are real roots of $G_2(\phi) = 0$ and $\phi_-^* < \phi_{23}$.

5). Under condition H5. The solitary wave solutions of Eq.(2) have expressions $u_5(x,t) = \phi_5(\xi)$ and $u_6(x,t) = \phi_6(\xi)$.

$$\phi_5(\xi) = \phi_{02} - \frac{2a_3}{b_3 + (\phi_{23} - \phi_{21})\cosh(e_3\xi)}. \tag{20}$$

$$\phi_6(\xi) = \phi_{02} + \frac{2a_3}{-b_3 + (\phi_{23} - \phi_{21})\cosh(e_3\xi)}, \tag{21}$$

Where $a_3 = (\phi_{02} - \phi_{21})(\phi_{23} - \phi_{02})$, $b_3 = \phi_{21} + \phi_{23} - 2\phi_{02}$, $e_3 = \sqrt{\frac{\beta}{2\gamma}a_3}$, and the $(\phi_{02}, 0)$ is saddle point, ϕ_{21}, ϕ_{02} and ϕ_{23} are real roots of $G_2(\phi) = 0$, $\phi_{21} < \phi_{02} < \phi_{23}$.

Proof. We prove the 1) and 3) only. The other case can use a similar argument.

1). Under condition H1, namely $\beta < 0$, $c + \alpha > 0$ and $-g_- < g < 0$. The $(\phi_{01}, 0)$ and $(\phi_{03}, 0)$ are saddle points of the system (5), the $(\phi_{02}, 0)$ is center point of The system (5), $\phi_{01} < \phi_{02} < \phi_{03}$. the system (5) has the one homoclinic orbit Γ_2 passing through the $(\phi_{01}, 0)$ (Fig.2.(c)). Therefore the Eq.(2) has one solitary wave.

Obviously, the $G_1(\phi) = 0$ has the root ϕ_{01} with multiple 2, the single root ϕ_{12} and ϕ_{13} satisfying $\phi_{01} < \phi_{12} < \phi_{13}$. Therefore we have

$$G_1(\phi) = -\frac{\beta}{4\gamma}(\phi - \phi_{01})^2(\phi - \phi_{12})(\phi - \phi_{13}), \; for \; \phi_{01} \le \phi \le \phi_{12}, \tag{22}$$

$$y = \pm(\phi - \phi_{01})\sqrt{-\frac{\beta}{2\gamma}(\phi_{12} - \phi)(\phi_{13} - \phi)}, \; for \; \phi_{01} \le \phi \le \phi_{12}. \tag{23}$$

Substituting (23) into $\frac{d\phi}{d\xi} = y$ and integrating it along Γ_2, we can obtain (16).

3). Under condition H3, namely $\beta > 0$, $c + \alpha < 0$ and $g = -g_+$. The $(\phi_{01}, 0)$ is center point of the system (5), the $(\phi_{02}, 0)$ is degenerate saddle point of the system (5), $\phi_{01} < \phi_{02} = \phi_+^*$. the system (5) has the one homoclinic orbit Γ_1 passing through the $(\phi_+^*, 0)$ (Fig.2.(h)). Therefore the Eq.(2) has one solitary wave.

Obviously, the $G_2(\phi) = 0$ has the root ϕ_+^* with multiple 3, the single root ϕ_{21} , satisfying $\phi_{21} < \phi_+^*$. Therefore we have

$$G_2(\phi) = -\frac{\beta}{4\gamma}(\phi - \phi_+^*)^3(\phi - \phi_{21}), \; for \; \phi_{21} \le \phi \le \phi_+^*, \tag{24}$$

$$y = \pm(\phi_+^* - \phi)\sqrt{\frac{\beta}{2\gamma}(\phi_+^* - \phi)(\phi - \phi_{21})}, \; for \; \phi_{21} \le \phi \le \phi_+^*. \tag{25}$$

Substituting (25) into $\frac{d\phi}{d\xi} = y$ and integrating it along Γ_1, we can obtain (18).

Remark 2. From (16)-(21), the planar graphs of the solitary waves Eq.(2) can be simulated by use mathematical software *Maple.* For example, when $\alpha = -5, \beta = 1, c = 0.5, \gamma = 2$ and $g = 2$, from (20) and (21), the numerical simulation results are Fig.3 (d) and (h).

5 Conclusion

In this paper, we have studied the bifurcation and global behavior a GIBE, and obtained the condition under which the solitary wave solutions appear. The planar graphs of the periodic waves and solitary waves are simulated in Fig.3. Finally, the representations of the solitary waves be obtained. Numerical simulation results show the consistence with the theoretical analysis.

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