On Certain Subclasses of Analytic Functions Defined by a Multiplier Transformation with Two Parameters

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Abstract

Let $A$ denote the class of analytic functions with the normalization $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$, set

$$f_n^b,\lambda(z) = z + \sum_{k=2}^{\infty} \left( \frac{k+b}{1+b} \right) \frac{n(k+\lambda-1)!}{\lambda!(k-1)!} z^k$$

($n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}, \lambda > -1; z \in \mathbb{U}$).

and define $(f_n^b,\lambda)^{(-1)}$ in terms of the Hadamard product

$$f_n^b,\lambda(z) \ast (f_n^b,\lambda)^{(-1)}(z) = \frac{z}{(1-z)^\mu} \quad (\mu > 0; z \in \mathbb{U}).$$

In this paper, the authors introduce several new subclasses of analytic functions defined by means of the operator $I_n^b,\lambda,\mu : A \rightarrow A$, given by

$$I_n^b,\lambda,\mu f(z) = (f_n^b,\lambda)^{(-1)} \ast f(z) \quad (f \in A; n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}, \lambda > -1; \mu > 0).$$

Inclusion properties of these classes and the classes involving the generalized Libera integral operator are also considered

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1 Introduction

Let \( \mathcal{A} \) denote the class of functions of the form
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\] (1.1)
which are analytic in the unit disk \( \mathbb{U} = \{z : |z| < 1\} \). If \( f \) and \( g \) are analytic in \( \mathbb{U} \), we say that \( f \) is subordinate to \( g \), written \( f \prec g \) or \( f(z) \prec g(z) \), if there exists Schwarz function \( w \) in \( \mathbb{U} \) such that \( f(z) = g(w(z)) \). We denote by \( S^*(\gamma), C(\gamma) \) and \( K(\gamma, \delta) \) the subclasses of \( \mathcal{A} \) consisting of all analytic functions which are, respectively, starlike of order \( \gamma \) \((0 \leq \gamma < 1)\), convex of order \( \gamma \) \((0 \leq \gamma < 1)\) and close-to-convex of order \( \delta \) type \( \gamma \) in \( \mathbb{U} \).(see, e.g., Srivastava and Owa [1]).

For \( n \in \mathbb{C} \), \( b \in \mathbb{C} \setminus \mathbb{Z}^- \) and \( \lambda > -1 \), the authors [4] introduced the Multiplier transformation \( D^n_{b,\lambda} \) of functions \( f \in \mathcal{A} \) by
\[
D^n_{b,\lambda} f(z) = z + \sum_{k=2}^{\infty} \left( \frac{k+b}{1+b} \right) \frac{n(k+\lambda-1)!}{\lambda!(k-1)!} a_k z^k.
\]

Let \( \mathcal{P} \) be the class of all functions \( \phi \) which are analytic and univalent in \( \mathbb{U} \) and for which \( \phi(\mathbb{U}) \) is convex with \( \phi(0) = 1 \) and \( \Re \{\phi(z)\} > 0 \) for \( z \in \mathbb{U} \).

Making use of subordination principle between two analytic functions, we introduce the subclasses \( S^*(\gamma; \phi), C(\gamma; \phi) \) and \( K(\gamma, \delta; \phi, \psi) \) of the class \( \mathcal{A} \) for \( 0 \leq \gamma, \delta < 1 \) and \( \phi, \psi \in \mathcal{P} \) (cf., [2]), which are defined by

\[
S^*(\gamma; \phi) = \left\{ f : f \in \mathcal{A} \text{ and } \frac{1}{1-\gamma} \left( \frac{zf'(z)}{f(z)} - \gamma \right) < \phi(z) \text{ in } \mathbb{U} \right\},
\]
\[
C(\gamma; \phi) = \left\{ f : f \in \mathcal{A} \text{ and } \frac{1}{1-\gamma} \left( 1 + \frac{zf''(z)}{f''(z)} - \gamma \right) < \phi(z) \text{ in } \mathbb{U} \right\},
\]
and
\[
K(\gamma, \delta; \phi, \psi) = \left\{ f : f \in \mathcal{A} \text{ and } \exists g \in S^*(\gamma; \phi) \text{ s.t.} \frac{1}{1-\delta} \left( \frac{zf'(z)}{g(z)} - \delta \right) < \psi(z) \text{ in } \mathbb{U} \right\}.
\]
In particular, when $\gamma = \delta = 0$ we have the classes $S^*(\phi), C(\phi),$ and $K(\phi, \psi)$ investigated by Ma and Minda $[12]$ and Kim et al. $[13]$. For suitable choices of $\phi$ and $\psi$ we can easily gather the various subclasses of $A$. For example:

$$S^*(\gamma; \frac{1+z}{1-z}) = S^*(\gamma), \quad C(\gamma; \frac{1+z}{1-z}) = C(\gamma),$$

$$K(\gamma, \delta; \frac{1+z}{1-z}, \frac{1+z}{1-z}) = K(\gamma, \delta).$$

Setting

$$f_{b,\lambda}^n(z) = z + \sum_{k=2}^{\infty} \frac{(k+b)^{n(k+\lambda-1)!}}{1+b} \frac{\lambda(k-1)!}{\lambda(k-1)!} z^k\quad (n \in \mathbb{C}, \ b \in \mathbb{C} \setminus \mathbb{Z}^-),$$

we define a new function $(f_{b,\lambda}^n)^{-1}(z)$ in terms of the Hadamard product (or convolution)

$$(f_{b,\lambda}^n(z) * (f_{b,\lambda}^n)^{-1}(z) = \frac{z}{(1-z)^\mu}\quad (\mu > 0; \ z \in \mathbb{U}).$$

Then, motivated essentially by the Choi-Sagio-Srivastava operator $[2]$ (see also $[3]$ and $[5]$), we now introduce the operator $I_{b,\lambda,\mu}^n : A \rightarrow A$, which are defined here by

$$(I_{b,\lambda,\mu}^n f(z) = (f_{b,\lambda}^n)^{-1}(z) * f(z) \quad (1.2)$$

$$(f \in A; n \in \mathbb{C}, \ b \in \mathbb{C} \setminus \mathbb{Z}^-, \ \lambda > -1, \ \mu > 0).$$

Note that $I_{b,\lambda,\mu}^0$ the Choi-Sagio-Srivastava operator was introduced and studied by Choi et al. $[2]$, and $I_{\lambda,\mu,2}^0 (\lambda \in \mathbb{N}_0)$ the Noor integral operator was introduced by Noor $[5]$. (see also $[3]$). Also the operator $I_{b,0,\mu}^n (n \in \mathbb{R}; \ b > -1)$ was studied by Cho and Kim $[6]$. In particular, we note that $I_{0,0,2}^0 = zf'(z)$ and $I_{0,0,2}^0 = T_{0,0,2}^0 = f(z)$. In view of (1.1) and (1.2), for $n \in \mathbb{C}, \ b \in \mathbb{C} \setminus \mathbb{Z}^-, \ \lambda > -1, \ \mu > 0$, we obtain the following relations:

$$z(I_{b,\lambda,\mu}^{n+1} f(z))' = (b+1)I_{b,\lambda,\mu}^n f(z) - bI_{b,\lambda,\mu}^{n+1} f(z), \quad (1.3)$$

and

$$z(I_{b,\lambda+1,\mu}^{n} f(z))' = (\lambda+1)I_{b,\lambda,\mu}^n f(z) - \lambda I_{b,\lambda+1,\mu}^n f(z), \quad (1.4)$$

and

$$z(I_{b,\lambda,\mu}^{n} f(z))' = \mu I_{b,\lambda,\mu+1}^n f(z) - (\mu-1)I_{b,\lambda,\mu}^{n} f(z), \quad (1.5)$$
Next, by using the operator $I_{n}^{*}$, we introduce the following classes of analytic functions for $\phi, \psi \in \mathcal{P}, n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}^{-}, \lambda > -1, \mu > 0$ and $0 \leq \gamma, \delta < 1$:

\begin{align*}
S_{n}^{a}(\gamma; \phi) & = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad I_{n}^{*}f(z) \in S^{*}(\gamma; \phi) \right\}, \\
C_{n}^{a}(\gamma; \phi) & = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad I_{n}^{*}f(z) \in \mathcal{C}(\gamma; \phi) \right\}, \\
K_{n}^{a}(\gamma; \delta; \phi, \psi) & = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad I_{n}^{*}f(z) \in K(\gamma, \delta; \phi, \psi) \right\}.
\end{align*}

We note that

\begin{equation}
\label{1.6}
f(z) \in C_{n}^{a}(\gamma; \phi) \Leftrightarrow zf'(z) \in S_{n}^{a}(\gamma; \phi).
\end{equation}

In particular, we set

\begin{align*}
S_{n}^{a}(\gamma, (1+Az)/(1+Bz); \alpha) & = S_{n}^{a}(\gamma; A, B; \alpha), \quad (0 < \alpha \leq 1; -1 \leq B < A \leq 1), \\
C_{n}^{a}(\gamma, (1+Az)/(1+Bz); \alpha) & = C_{n}^{a}(\gamma; A, B; \alpha), \quad (0 < \alpha \leq 1; -1 \leq B < A \leq 1).
\end{align*}

In this paper, we investigate several inclusion properties for the classes $S_{n}^{a}(\gamma; \phi), C_{n}^{a}(\gamma; \phi)$ and $K_{n}^{a}(\gamma, \delta; \phi, \psi)$ associated with the operator $I_{n}^{*}$.

Some applications involving these and other families of operator are also obtained.

**2 Inclusion properties involving $I_{n}^{*}$**

To derive our results we need the following lemmas:

**Lemma 2.1** [7]. Let $\beta, \nu$ be complex numbers. Let $\phi \in \mathcal{P}$ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $\Re \left[ \beta \phi(z) + \nu \right] > 0, z \in \mathbb{U}$. If $p(z) = 1 + p_{1}z + p_{2}z^{2} + \cdots$ is analytic in $\mathbb{U}$ with $p(0) = 1$, then

\[ p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec \phi(z) \Rightarrow p(z) \prec \phi(z), \quad (z \in \mathbb{U}). \]

**Lemma 2.2** [11]. Let $\phi \in \mathcal{P}$ be convex univalent in $\mathbb{U}$ and $w$ be analytic in $\mathbb{U}$ with $\Re w(z) \geq 0, z \in \mathbb{U}$. If $p$ is analytic in $\mathbb{U}$ with $p(0) = \phi(0)$, then

\[ p(z) + w(z)p'(z) \prec \phi(z) \Rightarrow p(z) \prec \phi(z), \quad (z \in \mathbb{U}). \]
At first, with the help of Lemma 2.1, we obtain the following:

**Theorem 2.3**

\[ S^n_{b,\lambda,\mu+1}(\gamma; \phi) \subset S^n_{b,\lambda,\mu}(\gamma; \phi) \subset S^n_{b,\lambda,\mu}(\gamma; \phi), \]

for \( n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}^-, \lambda \geq 0, \mu \geq 1, 0 \leq \gamma < 1 \) and \( \phi \in \mathcal{P} \).

**Proof.** First, we will show that

\[ S^n_{b,\lambda,\mu+1}(\gamma; \phi) \subset S^n_{b,\lambda,\mu}(\gamma; \phi). \]

Let \( f \in S^n_{b,\lambda,\mu+1}(\gamma; \phi) \) and set

\[ p(z) = \frac{1}{1 - \gamma} \left( \frac{z(T^n_{b,\lambda,\mu}f(z))'}{T^n_{b,\lambda,\mu}f(z)} - \gamma \right), \quad (2.1) \]

where \( p \) analytic in \( U \) with \( p(0) = 1 \). Applying (1.5) and (2.1)

\[ \frac{T^n_{b,\lambda,\mu+1}f(z)}{\mu T^n_{b,\lambda,\mu}f(z)} = (1 - \gamma)p(z) + \gamma + \mu - 1. \quad (2.2) \]

Taking the logarithmic differentiation on both sides of (2.2) and multiplying by \( z \), we have

\[ \frac{1}{1 - \gamma} \left( \frac{z(T^n_{b,\lambda,\mu+1}f(z))'}{T^n_{b,\lambda,\mu+1}f(z)} - \gamma \right) = p(z) + \frac{zp'(z)}{(1 - \gamma)p(z) + \gamma + \mu - 1} (z \in \mathbb{U}). \quad (2.3) \]

Applying Lemma 2.1 to (2.3), it follows that \( p \prec \phi \), that is \( f \in S^n_{b,\lambda,\mu}(\gamma; \phi) \).

To prove the second part, let \( f \in S^n_{b,\lambda,\mu}(\gamma; \phi) \) and put

\[ h(z) = \frac{1}{1 - \gamma} \left( \frac{z(T^n_{b,\lambda,\mu+1}f(z))'}{T^n_{b,\lambda,\mu+1}f(z)} - \gamma \right), \]

where \( h \) is analytic in \( U \) with \( h(0) = 1 \). Then, by using the arguments similar to those detailed above with (1.3), it follows that \( h \prec \phi \) in \( \mathbb{U} \), which implies that \( f \in S^n_{b,\lambda,\mu}(\gamma; \phi) \). Therefore, we complete the proof of Theorem 2.1.

**Theorem 2.4**

\[ S^n_{b,\lambda,\mu+1}(\gamma; \phi) \subset S^n_{b,\lambda,\mu}(\gamma; \phi) \subset S^n_{b,\lambda+1,\mu}(\gamma; \phi), \]

for \( n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}^-, \lambda \geq 0, \mu \geq 1, 0 \leq \gamma < 1 \) and \( \phi \in \mathcal{P} \).
Proof. We proved the first part in Theorem 2.3. To prove the second part let \( f \in S_{b,\lambda,\mu}^n(\gamma, \phi) \) and put
\[
p(z) = \frac{1}{1 - \gamma} \left( \frac{z(I_{b,\lambda+1,\mu}^n f(z))'}{I_{b,\lambda+1,\mu}^n f(z)} - \gamma \right),
\]
where \( p \) is analytic in \( U \) with \( p(0) = 1 \). Then, by using the arguments similar to those detailed in Theorem 2.3 with (1.4), it follows that \( p \prec \phi \) in \( U \), which implies that \( f \in S_{b,\lambda+1,\mu}^n(\gamma; \phi) \).

**Theorem 2.5** Let \( n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}^-, \lambda \geq 0 \) and \( \mu \geq 1 \). Then
\[
C_{b,\lambda,\mu+1}^n(\gamma; \phi) \subset C_{b,\lambda,\mu}^n(\gamma; \phi) \subset C_{b,\lambda,\mu+1}^{n+1}(\gamma; \phi) \quad (0 \leq \gamma < 1; \phi \in \mathcal{P}).
\]

Proof. Applying (1.6) and Theorem 2.3, we observe that
\[
f(z) \in C_{b,\lambda,\mu+1}^n(\gamma; \phi) \iff I_{b,\lambda,\mu+1}^n f(z) \in C(\gamma; \phi)
\]
\[
\iff z(I_{b,\lambda,\mu+1}^n f(z))' \in S^*(\gamma; \phi)
\]
\[
\iff I_{b,\lambda,\mu+1}^n(z f'(z)) \in S^*(\gamma; \phi)
\]
\[
\iff z f'(z) \in S_{b,\lambda,\mu}^n(\gamma; \phi)
\]
\[
\iff I_{b,\lambda,\mu}^n(z f'(z)) \in C(\gamma; \phi)
\]
\[
\iff z f(z) \in C_{b,\lambda,\mu}^n(\gamma; \phi),
\]
and
\[
f(z) \in C_{b,\lambda,\mu}^n(\gamma, \phi) \iff z f'(z) \in S_{b,\lambda,\mu}^n(\gamma; \phi)
\]
\[
\iff z f'(z) \in S_{b,\lambda,\mu+1}^n(\gamma; \phi)
\]
\[
\iff z(I_{b,\lambda,\mu}^{n+1} f(z))' \in S^*(\gamma; \phi)
\]
\[
\iff I_{b,\lambda,\mu}^{n+1} f(z) \in C(\gamma; \phi)
\]
\[
\iff f(z) \in C_{b,\lambda,\mu}^{n+1}(\gamma; \phi),
\]
which evidently proves Theorem 2.5.

By applying (1.6) and Theorem 2.4 and using the same methods to prove Theorem 2.5 we can prove the follow:

**Theorem 2.6** Let \( n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}^-, \lambda \geq 0 \) and \( \mu \geq 1 \). Then
\[
C_{b,\lambda,\mu+1}^n(\gamma; \phi) \subset C_{b,\lambda,\mu}^n(\gamma; \phi) \subset C_{b,\lambda,\mu+1}^n(\gamma; \phi) \quad (0 \leq \gamma < 1; \phi \in \mathcal{P}).
\]
Taking
\[ \phi(z) = \left( \frac{1 + Az}{1 + Bz} \right)^\alpha \quad (-1 \leq B < A \leq 1; 0 < \alpha \leq 1; z \in \mathbb{U}) \] (2.4)
in Theorem 2.3 and Theorem 2.5, we have the following

**Corollary 2.7** Let \( n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}^-, \lambda \geq 0 \) and \( \mu \geq 1 \). Then
\[ S_{b,\lambda,\mu+1}^n(\gamma; A, B; \alpha) \subset S_{b,\lambda,\mu}^n(\gamma; A, B; \alpha) \subset S_{b,\lambda,\mu+1}^{n+1}(\gamma; A, B; \alpha) \]
\[ (0 \leq \gamma < 1; -1 \leq B < A \leq 1; 0 < \alpha \leq 1), \]
and
\[ C_{b,\lambda,\mu+1}^n(\gamma; A, B; \alpha) \subset C_{b,\lambda,\mu}^n(\gamma; A, B; \alpha) \subset C_{b,\lambda,\mu+1}^{n+1}(\gamma; A, B; \alpha) \]
\[ (0 \leq \gamma < 1; -1 \leq B < A \leq 1; 0 < \alpha \leq 1). \]

Also, by taking (2.4) in Theorem 2.4 and Theorem 2.6, we have the following

**Corollary 2.8** Let \( n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}^-, \lambda \geq 0 \) and \( \mu \geq 1 \). Then
\[ S_{b,\lambda,\mu+1}^n(\gamma; A, B; \alpha) \subset S_{b,\lambda,\mu}^n(\gamma; A, B; \alpha) \subset S_{b,\lambda+1,\mu}^n(\gamma; A, B; \alpha) \]
\[ (0 \leq \gamma < 1; -1 \leq B < A \leq 1; 0 < \alpha \leq 1), \]
and
\[ C_{b,\lambda,\mu+1}^n(\gamma; A, B; \alpha) \subset C_{b,\lambda,\mu}^n(\gamma; A, B; \alpha) \subset C_{b,\lambda+1,\mu}^n(\gamma; A, B; \alpha) \]
\[ (0 \leq \gamma < 1; -1 \leq B < A \leq 1; 0 < \alpha \leq 1). \]

Next, by using Lemma 2.2, we obtain the following inclusion relation for the class \( K_{b,\lambda,\mu}^n(\gamma, \delta; \phi, \psi) \).

**Theorem 2.9** Let \( n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}^-, \lambda > -1 \) and \( \mu \geq 1 \). Then
\[ K_{b,\lambda,\mu+1}^n(\gamma, \delta; \phi, \psi) \subset K_{b,\lambda,\mu}^n(\gamma, \delta; \phi, \psi) \subset K_{b,\lambda,\mu+1}^{n+1}(\gamma, \delta; \phi, \psi) \]
\[ (0 \leq \gamma, \delta < 1; \phi, \psi \in \mathcal{P}). \]

**Proof.** We begin by proving that
\[ K_{b,\lambda,\mu+1}^n(\gamma, \delta; \phi, \psi) \subset K_{b,\lambda,\mu}^n(\gamma, \delta; \phi, \psi). \]

Let \( f \in K_{b,\lambda,\mu+1}^n(\gamma, \delta; \phi, \psi) \). Then, in view of the definition of the class \( K_{b,\lambda,\mu+1}^n(\gamma, \delta; \phi, \psi) \), there exists a function \( g \in S_{b,\lambda,\mu+1}^n(\gamma; \phi) \) such that
\[ \frac{1}{1 - \delta} \left( \frac{z(I_{b,\lambda,\mu+1}^n f(z))'}{I_{b,\lambda,\mu+1}^n g(z)} - \delta \right) \prec \psi(z) \quad (z \in \mathbb{U}). \]
Now let
\[ p(z) = \frac{1}{1 - \delta} \left( \frac{z(I_{b,\lambda,\mu}^n f(z))'}{I_{b,\lambda,\mu}^n g(z)} - \delta \right) \]
where \( p \) is analytic in \( U \) with \( p(0) = 1 \). Using (1.5), we obtain
\[ [(1 - \delta)p(z) + \delta I_{b,\lambda,\mu}^n g(z) + (\mu - 1)I_{b,\lambda,\mu}^n f(z) = \mu I_{b,\lambda,\mu+1}^n f(z). \]  
(2.5)

Differentiating (2.5) and multiplying by \( z \), we have
\[ (1 - \delta)zp'(z)I_{b,\lambda,\mu}^n g(z) + [(1 - \delta)p(z) + \delta z(I_{b,\lambda,\mu}^n g(z))'] = \mu z(I_{b,\lambda,\mu+1}^n f(z)') - (\mu - 1)z(I_{b,\lambda,\mu}^n f(z)'). \]  
(2.6)

Since \( g \in S_{b,\lambda,\mu+1}^n(\gamma; \phi) \), by Theorem 2.3, we know that \( g \in S_{b,\lambda,\mu}^n(\gamma; \phi) \). Let
\[ q(z) = \frac{1}{1 - \gamma} \left( \frac{z(I_{b,\lambda,\mu}^n g(z))'}{I_{b,\lambda,\mu}^n g(z)} - \gamma \right). \]

Then, using (1.5) once again, we have
\[ \frac{I_{b,\lambda,\mu+1}^n g(z)}{I_{b,\lambda,\mu}^n g(z)} = (1 - \gamma)q(z) + \mu + \gamma - 1. \]  
(2.7)

From (2.6) and (2.7), we obtain
\[ \frac{1}{1 - \delta} \left( \frac{z(I_{b,\lambda,\mu+1}^n f(z))'}{I_{b,\lambda,\mu+1}^n g(z)} - \delta \right) = \frac{zp'(z)}{(1 - \gamma)q(z) + \mu + \gamma - 1}. \]

Since \( \mu \geq 1 \) and \( q \prec \phi \) in \( U \),
\[ \text{Re} \{ (1 - \gamma)q(z) + \mu + \gamma - 1 \} > 0 \quad (z \in U). \]

Hence applying Lemma 2.2, we can show that \( p \prec \phi \), so that \( f \in K_{b,\lambda,\mu}^n(\gamma, \delta; \phi, \psi) \). For the second part, by using the arguments similar to those detailed above with (1.3), we obtain
\[ K_{b,\lambda,\mu}^n(\gamma, \delta; \phi, \psi) \subset K_{b,\lambda,\mu}^{n+1}(\gamma, \delta; \phi, \psi). \]

Therefore, we complete the proof of Theorem 2.9.

**Theorem 2.10** Let \( n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}^-, \lambda \geq 0 \) and \( \mu \geq 1 \). Then
\[ K_{b,\lambda,\mu+1}^n(\gamma, \delta; \phi, \psi) \subset K_{b,\lambda,\mu}^n(\gamma, \delta; \phi, \psi) \subset K_{b,\lambda,\mu+1}^n(\gamma, \delta; \phi, \psi) \]
\[ (0 \leq \gamma, \delta < 1; \phi, \psi \in \mathcal{P}). \]

**Proof.** We proved the first part in Theorem 2.9. For the second part, by using the arguments similar to those detailed in Theorem 2.9 with (1.4).
3 Inclusion properties involving the integral operator $F_c$

In this section, we consider the generalized Libera integral operator $F_c$ [8] (see also [9] and [10]) defined by

$$F_c(z) = F_c(f)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) \, dt \quad (f \in \mathcal{A}; c > -1). \quad (3.1)$$

From the definition of $F_c$ defined by (3.1), we observe that

$$z(T_{b,\lambda,\mu}^n F_c(z))' = (c + 1)T_{b,\lambda,\mu}^n f(z) - cT_{b,\lambda,\mu}^n F_c(z). \quad (3.2)$$

**Theorem 3.1** Let $c \geq 0$, $n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$, $\lambda > -1$ and $\mu > 0$. If $f \in \mathcal{S}_{b,\lambda,\mu}^n (\gamma; \phi)$, $(0 \leq \gamma < 1; \phi \in \mathcal{P})$, then $F_c(z) \in \mathcal{S}_{b,\lambda,\mu}^n (\gamma; \phi)$, $(0 \leq \gamma < 1; \phi \in \mathcal{P})$.

**Proof.** Let $f \in \mathcal{S}_{b,\lambda,\mu}^n (\gamma; \phi)$ and set

$$p(z) = \frac{1}{1 - \gamma} \left( \frac{z(T_{b,\lambda,\mu}^n F_c(z))'}{T_{b,\lambda,\mu}^n F_c(z)} - \gamma \right), \quad (3.3)$$

where $p$ is analytic in $\mathbb{U}$ with $p(0) = 1$.

By using (3.2) and (3.3), we obtain

$$(c + 1) \frac{T_{b,\lambda,\mu}^n f(z)}{T_{b,\lambda,\mu}^n F_c(z)} = (1 - \gamma)p(z) + c + \gamma \quad (3.4)$$

Taking the logarithmic differentiation on both sides of (3.4) and multiplying by $z$, we have

$$\frac{1}{1 - \gamma} \left( \frac{z(T_{b,\lambda,\mu}^n f(z))'}{T_{b,\lambda,\mu}^n f(z)} - \gamma \right) = p(z) + \frac{zp'(z)}{(1 - \gamma)p(z) + c + \gamma} \quad (z \in \mathbb{U}).$$

Hence, by virtue of Lemma 2.1, we conclude that $p \prec \phi$ in $\mathbb{U}$, which implies that $F_c(z) \in \mathcal{S}_{b,\lambda,\mu}^n (\gamma; \phi)$.

Next, we derive an inclusion property involving $F_c$, which is given by the following.

**Theorem 3.2** Let $c \geq 0$, $n \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$, $\lambda > -1$ and $\mu > 0$. If $f \in \mathcal{C}_{b,\lambda,\mu}^n (\gamma; \phi)$, $(0 \leq \gamma < 1; \phi \in \mathcal{P})$, then $F_c(z) \in \mathcal{C}_{b,\lambda,\mu}^n (\gamma; \phi)$, $(0 \leq \gamma < 1; \phi \in \mathcal{P})$. 
Proof. By applying Theorem 3.1, it follows that

\[
\begin{align*}
    f(z) \in C_{b,\lambda,\mu}^n(\gamma; \phi) &\iff zf'(z) \in S_{b,\lambda,\mu}^n(\gamma; \phi) \\
    &\iff F_c(zf'(z)) \in S_{b,\lambda,\mu}^n(\gamma; \phi) \\
    &\iff z(F_c(z))' \in S_{b,\lambda,\mu}^n(\gamma; \phi) \\
    &\iff F_c(z) \in C_{b,\lambda,\mu}^n(\gamma; \phi),
\end{align*}
\]

which proves Theorem 3.2.

From Theorem 3.1 and Theorem 3.2, we have the following.

**Corollary 3.3** Let \( c \geq 0, n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}^- \), \( \lambda > -1 \) and \( \mu > 0 \). If \( f \) belongs to the class \( S_{b,\lambda,\mu}^n(\gamma; A, B; \alpha) \) (or \( C_{b,\lambda,\mu}^n(\gamma; A, B; \alpha) \)) \((0 \leq \gamma < 1; -1 \leq B < A \leq 1; 0 < \alpha \leq 1)\), the \( F_c(z) \) belongs to the class \( S_{b,\lambda,\mu}^n(\gamma; A, B; \alpha) \) (or \( C_{b,\lambda,\mu}^n(\gamma; A, B; \alpha) \)) \((0 \leq \gamma < 1; -1 \leq B < A \leq 1; 0 < \alpha \leq 1)\).

Finally, we prove the following.

**Theorem 3.4** \( c \geq 0, n \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}^- \), \( \lambda > -1 \) and \( \mu > 0 \). If \( f \in K_{b,\lambda,\mu}^n(\gamma, \delta; \phi, \psi) \), \((0 \leq \gamma < 1; \phi \in \mathcal{P})\), then \( F_c(z) \in K_{b,\lambda,\mu}^n(\gamma, \delta; \phi, \psi) \), \((0 \leq \gamma < 1; \phi \in \mathcal{P})\).

**Proof.** Let \( f \in K_{b,\lambda,\mu}^n(\gamma, \delta; \phi, \psi) \). Then, in view of the definition of the class \( K_{b,\lambda,\mu}^n(\gamma, \delta; \phi, \psi) \), there exists a function \( g \in S_{b,\lambda,\mu}^n(\gamma; \phi) \) such that

\[
\frac{1}{1 - \delta} \left( \frac{z(T_{b,\lambda,\mu}^n f(z))'}{T_{b,\lambda,\mu}^n g(z)} - \frac{z(T_{b,\lambda,\mu}^n F_c(z))'}{T_{b,\lambda,\mu}^n F_c(g)(z)} \right) \prec \psi(z) \quad (z \in \mathbb{U}).
\]

Thus, we set

\[
p(z) = \frac{1}{1 - \delta} \left( \frac{z(T_{b,\lambda,\mu}^n f(z))'}{T_{b,\lambda,\mu}^n F_c(g)(z)} - \frac{z(T_{b,\lambda,\mu}^n F_c(z))'}{T_{b,\lambda,\mu}^n F_c(g)(z)} \right).
\]

where \( p \) is analytic in \( \mathbb{U} \) with \( p(0) = 1 \). Since \( g \in S_{b,\lambda,\mu}^n(\gamma; \phi) \), we see from Theorem 3.1 that \( F_c(g) \in S_{b,\lambda,\mu}^n(\gamma; \phi) \). Using (3.2), we have

\[
(c + 1) \frac{z(T_{b,\lambda,\mu}^n f(z))'}{T_{b,\lambda,\mu}^n F_c(z)} = [(1 - \delta)p(z) + \delta][(1 - \gamma)q(z) + c + \gamma] + (1 - \delta)zp'(z),
\]

where

\[
q(z) = \frac{1}{1 - \gamma} \left( \frac{z(T_{b,\lambda,\mu}^n F_c(g)(z))'}{T_{b,\lambda,\mu}^n F_c(g)(z)} - \gamma \right).
\]
Hence, we have
\[
\frac{1}{1 - \delta} \left( z \frac{(T_{b,\lambda,\mu} f(z))'}{I_{b,\lambda,\mu} g(z)} - \delta \right) = p(z) + \frac{zp'(z)}{(1 - \gamma)q(z) + c + \gamma}.
\]

The remaining part of the proof in Theorem 3.4 is similar to that of Theorem 2.9 and so we omit it.

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References


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