Application of Hölder Inequality in Generalised Convolutions for Functions with Respect to $k$-Symmetric Points

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Abstract

Two classes of univalent functions with respect to $k$-symmetric points define on the unit disk satisfying the conditions:

$$\sum_{n=1}^{\infty} (n^k + 1 - \alpha)|a_{nk+1}| + \sum_{n=2; n\neq lk+1}^{\infty} n|a_n| \leq 1 - \alpha,$$

and

$$\sum_{n=1}^{\infty} (nk + 1)(n^k + 1 - \alpha)|a_{nk+1}| + \sum_{n=2; n\neq lk+1}^{\infty} n^2|a_n| \leq 1 - \alpha$$

are given. The two inequalities of the functions belonging to these two classes are the starlike and convex functions with respect to $k$-symmetric points, respectively. Some interesting properties of generalisations of Hadamard product in these classes are given.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$
which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $U$. Also let $\mathcal{T}$ denote the subclasses of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).$$

We denote by $S^*(\alpha)$ and $C(\alpha)$ for $0 \leq \alpha < 1$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike and convex functions of order $\alpha$. Thus by definition, we have

$$S^*(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (0 \leq \alpha < 1; z \in U) \right\},$$

and

$$C(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (0 \leq \alpha < 1; z \in U) \right\}.$$  

Also, denote by $\mathcal{T}S^*(\alpha)$ and $\mathcal{T}C(\alpha)$ the subclasses of $\mathcal{T}$ where

$$\mathcal{T}S^*(\alpha) = S^*(\alpha) \cap \mathcal{T} \quad \text{and} \quad \mathcal{T}C(\alpha) = C(\alpha) \cap \mathcal{T}.$$

Let $f_j(z) \in \mathcal{A}$, $(j = 1, 2, \cdots, m)$ be given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n.$$

Then the Hadamard product (or convolution) is defined by:

$$f_1(z) \ast f_2(z) \ast \cdots \ast f_m(z) = (f_1 \ast f_2 \ast \cdots \ast f_m)(z) = z + \sum_{n=2}^{\infty} \left( \prod_{j=1}^{m} a_{n,j} \right) z^n.$$

Also the generalised Hadamard Product is defined here by

$$(f_1 \diamond f_2 \diamond \cdots \diamond f_m)(z) = z + \sum_{n=2}^{\infty} \left( \prod_{j=1}^{m} (a_{n,j})^{1/p_j} \right) z^n.$$

where $\sum_{j=1}^{m} \frac{1}{p_j} = 1$, $p_j > 1$ and $j = 1, 2, \cdots m$.  

Let $F_j(z) \in T (j = 1, 2, \ldots, m)$ be given by

$$F_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad (a_{n,j} \geq 0).$$

Then the modified Hadamard product is defined by

$$(F_1 \ast F_2 \ast \cdots \ast F_m)(z) = z - \sum_{n=2}^{\infty} \left( \prod_{j=1}^{m} a_{n,j} \right) z^n \quad (a_{n,j} \geq 0).$$

Also the generalised modified Hadamard Product is defined here by

$$(F_1 \circ F_2 \circ \cdots \circ F_m)(z) = z - \sum_{n=2}^{\infty} \left( \prod_{j=1}^{m} \left( a_{n,j} \right)^{\frac{1}{p_j}} \right) z^n \quad (a_{n,j} \geq 0).$$

where $\sum_{j=1}^{m} \frac{1}{p_j} = 1, p_j > 1$ and $j = 1, 2, \ldots, m$.

Sakaguchi [1] once introduced a class $S_\ast^S$ of functions starlike with respect to symmetric points, which consists of functions $f \in S$ satisfying the inequality

$$\text{Re} \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in \mathbb{U}.$$ 

Many different authors have studied the work of Sakaguchi [1] and have discussed extensively about this class and its subclasses (see [2-9]). In 1979 Chand and Singh [5] introduced the classes $S_{\ast}^{(k)}(\alpha)$ of functions starlike with respect to $k$-symmetric points of order $\alpha$, and $C_{\ast}^{(k)}(\alpha)$ of functions convex with respect to $k$-symmetric points of order $\alpha$ which are the special classes corresponding to the ones defined in [9], which satisfy the following:

$$S_{\ast}^{(k)}(\alpha) = \left\{ f : f \in S \text{ and } \text{Re} \left( \frac{zf'(z)}{f_k(z)} \right) > \alpha \quad (0 \leq \alpha < 1; z \in \mathbb{U}) \right\},$$

and

$$C_{\ast}^{(k)}(\alpha) = \left\{ f : f \in S \text{ and } \text{Re} \left( \frac{(zf'(z))'}{f_k(z)} \right) > \alpha \quad (0 \leq \alpha < 1; z \in \mathbb{U}) \right\},$$

where $k \geq 1$ is a positive integer and $f_k(z)$ is defined by the following equality

$$f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^\nu z), \quad (\varepsilon = \exp(2\pi i/k); z \in \mathbb{U}).$$
Note that the function $f(z) \in A$ is in the class $C_S^{(k)}(\alpha)$ if and only if $zf'(z) \in S_S^{(k)}(\alpha)$.

Finally, denote by $\mathcal{T}S_S^{(k)}(\alpha)$ and $\mathcal{T}C_S^{(k)}(\alpha)$ the subclasses of $\mathcal{T}$ where

$$\mathcal{T}S_S^{(k)}(\alpha) = S_S^{(k)}(\alpha) \cap \mathcal{T} \quad \text{and} \quad \mathcal{T}C_S^{(k)}(\alpha) = C_S^{(k)}(\alpha) \cap \mathcal{T}.$$  

Now we state the results due to [9] as a special case when $\lambda = 0$, which we will use throughout this paper.

**Theorem 1.1** Let $0 \leq \alpha < 1$, $k \geq 1$. If

$$\sum_{n=1}^{\infty} (nk + 1 - \alpha)|a_{nk+1}| + \sum_{n=2}^{\infty} n|a_n| \leq 1 - \alpha, \quad \text{(1.1)}$$

then $f(z) \in S_S^{(k)}(\alpha)$. Condition (1.1) is also necessary if $f(z) \in \mathcal{T}S_S^{(k)}(\alpha)$.

**Theorem 1.2** Let $0 \leq \alpha < 1$, $k \geq 1$. If

$$\sum_{n=1}^{\infty} (nk + 1)(nk + 1 - \alpha)|a_{nk+1}| + \sum_{n=2}^{\infty} n^2|a_n| \leq 1 - \alpha, \quad \text{(1.2)}$$

then $f(z) \in C_S^{(k)}(\alpha)$. Condition (1.2) is also necessary if $f(z) \in \mathcal{T}C_S^{(k)}(\alpha)$.

In the present paper, we shall make use of the generalised Hadamard product with a view of Theorems 1.1 and 1.2 to prove interesting characterisation theorems involving the classes $S_S^{(k)}(\alpha)$, $C_S^{(k)}(\alpha)$, $\mathcal{T}S_S^{(k)}(\alpha)$ and $\mathcal{T}C_S^{(k)}(\alpha)$.

## 2 Generalised convolution properties of functions in the classes $S_S^{(k)}(\alpha)$, $\mathcal{T}S_S^{(k)}(\alpha)$

We state our first theorem as follows:

**Theorem 2.1** If $f_j \in S_S^{(k)}(\alpha_j)$, $(j = 1, 2, \cdots m)$, then

$$(f_1 \odot f_2 \odot \cdots \odot f_m)(z) \in S_S^{(k)}(\beta, \mu),$$
where
\[
\beta \leq \min_{n \geq 2} \left\{ 1 - \frac{nk}{\prod_{j=1}^{m} \left( \frac{nk+1-\alpha_j}{1-\alpha_j} \right)^{\frac{1}{p_j}} - 1} \right\},
\]
and
\[
\mu \leq \min_{\substack{n \geq 2 \\, n \neq lk+1}} \left\{ 1 - \frac{n}{\prod_{j=1}^{m} \left( \frac{n}{1-\alpha_j} \right)^{\frac{1}{p_j}}} \right\}.
\]
for \(\sum_{j=1}^{m} \frac{1}{p_j} = 1\), \(p_j > 1\).

**Proof.** Let \(f_j(z) \in S^{(k)}_S(\alpha_j)\), by using Theorem 1.1 we have:
\[
\sum_{n=1}^{\infty} \frac{nk+1-\alpha_j}{1-\alpha_j} |a_{nk+1,j}| + \sum_{n=2}^{\infty} \frac{n}{1-\alpha_j} |a_{n,j}| \leq 1, \quad (j = 1, 2, \cdots m).
\]

Moreover,
\[
\prod_{j=1}^{m} \left( \sum_{n=1}^{\infty} \left\{ \left( \frac{nk+1-\alpha_j}{1-\alpha_j} \right)^{\frac{1}{p_j}} |a_{nk+1,j}|^{\frac{1}{p_j}} \right\} \right)^{\frac{1}{p_j}}
+ \prod_{j=1}^{m} \left( \sum_{n=2}^{\infty} \left\{ \left( \frac{n}{1-\alpha_j} \right)^{\frac{1}{p_j}} |a_{n,j}|^{\frac{1}{p_j}} \right\} \right)^{\frac{1}{p_j}} \leq 1.
\]

By using the Hölder inequality, we have
\[
\sum_{n=1}^{\infty} \left\{ \prod_{j=1}^{m} \left( \frac{nk+1-\alpha_j}{1-\alpha_j} \right)^{\frac{1}{p_j}} |a_{nk+1,j}|^{\frac{1}{p_j}} \right\}
\leq \prod_{j=1}^{m} \left( \sum_{n=1}^{\infty} \left\{ \left( \frac{nk+1-\alpha_j}{1-\alpha_j} \right)^{\frac{1}{p_j}} |a_{nk+1,j}|^{\frac{1}{p_j}} \right\} \right)^{\frac{1}{p_j}},
\]
and

\[
\sum_{n=2}^{\infty} \left\{ \prod_{j=1}^{m} \left( \frac{n}{1-\alpha_j} \right)^{\frac{1}{p_j}} |a_{n,j}|^{\frac{1}{p_j}} \right\} \left( \sum_{n=2}^{\infty} \left\{ \left( \frac{n}{1-\alpha_j} \right)^{\frac{1}{p_j}} |a_{n,j}|^{\frac{1}{p_j}} \right\} \right)^{\frac{1}{p_j}} \leq \prod_{j=1}^{m} \left( \sum_{n=2}^{\infty} \left\{ \left( \frac{n}{1-\alpha_j} \right)^{\frac{1}{p_j}} |a_{n,j}|^{\frac{1}{p_j}} \right\} \right)^{\frac{1}{p_j}}.
\]

Then, we have

\[
\sum_{n=1}^{\infty} \left\{ \prod_{j=1}^{m} \left( \frac{n_{k+1}-\alpha_j}{1-\alpha_j} \right)^{\frac{1}{p_j}} |a_{nk+1,j}|^{\frac{1}{p_j}} \right\} + \sum_{n=2}^{\infty} \left\{ \prod_{j=1}^{m} \left( \frac{n}{1-\alpha_j} \right)^{\frac{1}{p_j}} |a_{n,j}|^{\frac{1}{p_j}} \right\} \leq 1.
\]

Here, we see that

\[
\sum_{n=1}^{\infty} \left\{ \left( \frac{n_{k+1}-\beta}{1-\beta} \right)^{\frac{1}{p_j}} \prod_{j=1}^{m} |a_{nk+1,j}|^{\frac{1}{p_j}} \right\} + \sum_{n=2}^{\infty} \left\{ \left( \frac{n}{1-\mu} \right)^{\frac{1}{p_j}} \prod_{j=1}^{m} |a_{n,j}|^{\frac{1}{p_j}} \right\} \leq 1
\]

with

\[
\beta \leq \min_{n \geq 2} \left\{ 1 - \frac{nk}{\prod_{j=1}^{m} \left( \frac{n_{k+1}-\alpha_j}{1-\alpha_j} \right)^{\frac{1}{p_j}} - 1} \right\},
\]

and

\[
\mu \leq \min_{n \geq 2 \atop n \neq lk + 1} \left\{ 1 - \frac{n}{\prod_{j=1}^{m} \left( \frac{n}{1-\alpha_j} \right)^{\frac{1}{p_j}}} \right\}.
\]

Thus, by Theorem 1.1, the proof of Theorem 2.1 is complete.

Next, we obtain our first corollary.
Corollary 2.2 If $f_j(z) \in S_S^{(k)}(\alpha)$, $(j = 1, \cdots, m)$, then

$$(f_1 \circ f_2 \circ \cdots \circ f_m)(z) \in S_S^{(k)}(\alpha),$$

**Proof.** In view of Theorem 1.1, Corollary 2.2 follows readily from Theorem 2.1 for the special case when $\alpha_j = \alpha$.

Further, we obtain the following results:

**Theorem 2.3** If $F_j(z) \in T S_S^{(k)}(\alpha_j)$, $(j = 1, \cdots, m)$, then

$$(F_1 \circ F_2 \circ \cdots \circ F_m)(z) \in T S_S^{(k)}(\beta, \mu),$$

where $\beta$ and $\mu$ given by conditions in Theorem 2.1 and for $\sum_{j=1}^{m} \frac{1}{p_j} = 1$, $p_j > 1$.

**Proof.** By using the same technique as in the proof of Theorem 2.1, the required result is obtained.

**Theorem 2.4** Let the function $f_j(z) \in S_S^{(k)}(\alpha_j)$, $(j = 1, \cdots, m)$, and let $t_m(z)$ be defined by

$$t_m(z) = z + \sum_{n=1}^{\infty} \left( \sum_{j=1}^{m} (a_{nk+1,j})^p \right) z^n + \sum_{n=2}^{\infty} \left( \sum_{j=1}^{m} (a_{nj})^p \right) z^n. \quad (2.1)$$

Then

$$t_m(z) \in S_S^{(k)}(\delta, \gamma),$$

where

$$\delta = 1 - \frac{n k}{m \left( \frac{n k + 1 - \alpha_j}{1 - \alpha_j} \right)^{\frac{p}{1 - \alpha_j}}} - 1, \quad \gamma = 1 - \frac{n}{m \left( \frac{n}{1 - \alpha} \right)^p}$$

and

$$\left( \frac{n k + 1 - \alpha_j}{1 - \alpha_j} \right)^{\frac{p}{1 - \alpha_j}}; \left( \frac{n}{1 - \alpha} \right)^{\frac{p}{1 - \alpha}} \geq mn, \alpha = \min_{1 \leq j \leq m} \alpha_j.$$

**Proof.** Since $f_j \in S_S^{(k)}(\alpha_j)$, using Theorem 1.1, we observe that

$$\sum_{n=1}^{\infty} \left( \frac{n k + 1 - \alpha_j}{1 - \alpha_j} \right)^p |a_{nk+1,j}|^p + \sum_{n=2}^{\infty} \left( \frac{n}{1 - \alpha_j} \right)^p |a_{nj}|^p$$

$$\leq \left( \sum_{n=1}^{\infty} \frac{n k + 1 - \alpha_j}{1 - \alpha_j} |a_{nk+1,j}| \right)^p + \left( \sum_{n=2}^{\infty} \frac{n}{1 - \alpha_j} |a_{nj}| \right)^p \leq 1.$$  

(2.2)
It follows from (2.2) that
\[
\sum_{n=1}^{\infty} \left\{ \frac{1}{m} \sum_{j=1}^{m} \left( \frac{nk+1-\alpha_j}{1-\alpha_j} \right)^p |a_{nk+1,j}|^p \right\} + \sum_{n=2}^{\infty} \left\{ \frac{1}{m} \sum_{j=1}^{m} \left( \frac{n-\alpha_j}{1-\alpha_j} \right)^p |a_{n,j}|^p \right\} \leq 1.
\]

Putting \( \alpha = \min_{1 \leq j \leq m} \alpha_j \), and by virtue of Theorem 1.1, we find that
\[
\sum_{n=1}^{\infty} \frac{nk+1-\delta}{1-\delta} \sum_{j=1}^{m} |a_{nk+1,j}|^p + \sum_{n=2}^{\infty} \sum_{j=1}^{m} |a_{n,j}|^p \leq 1,
\]
if, \( \delta = 1 - \frac{nk}{\frac{1}{m} \left( \frac{nk+1-\alpha}{1-\alpha} \right)^p - 1} \), \( \gamma = 1 - \frac{n}{\frac{1}{m} \left( \frac{n}{1-\alpha} \right)^p} \).

Now let
\( u(n) = 1 - \frac{nk}{\frac{1}{m} \left( \frac{nk+1-\alpha}{1-\alpha} \right)^p - 1} \), \( v(n) = 1 - \frac{n}{\frac{1}{m} \left( \frac{n}{1-\alpha} \right)^p} \).

Then \( u'(n), v'(n) \geq 0 \) if \( p \geq 2 \). Hence
\( \delta \leq 1 - \frac{nk}{\frac{1}{m} \left( \frac{nk+1-\alpha}{1-\alpha} \right)^p - 1} \), \( \gamma \leq 1 - \frac{n}{\frac{1}{m} \left( \frac{n}{1-\alpha} \right)^p} \).

By \( \left( \frac{nk+1-\alpha}{1-\alpha} \right)^p, \left( \frac{n}{1-\alpha} \right)^p \geq mn \), we see that \( 0 \leq \delta < 1 \) and \( 0 \leq \gamma < 1 \).

Thus the proof of Theorem 2.4 is complete.
3 Generalised convolution properties of functions in the classes $C_S^{(k)}(\alpha)$, $TC_S^{(k)}(\alpha)$

In this section, we give another set of results regarding the classes $C_S^{(k)}(\alpha)$ and $TC_S^{(k)}(\alpha)$.

**Theorem 3.1** If the functions $f_j \in C_S^{(k)}(\alpha_j)$, $(j = 1, \cdots, m)$, then

$$(f_1 \ast f_2 \ast \cdots \ast f_m)(z) \in C_S^{(k)}(\beta, \mu),$$

where $\beta$ and $\mu$ given by conditions in Theorem 2.1 and for $\sum_{j=1}^{m} \frac{1}{p_j} = 1$, $p_j > 1$.

**Proof.** Let $f_j \in C_S^{(k)}(\alpha_j)(j = 1, \cdots, m)$, by using Theorem 1.2, we have

$$\sum_{n=1}^{\infty} \frac{(nk + 1)(nk + 1 - \alpha_j)}{1 - \alpha_j} |a_{nk+1,j}| + \sum_{n=2}^{\infty} \frac{n^2}{1 - \alpha_j} |a_{n,j}| \leq 1.$$

Thus the proof of Theorem 3.1 is much akin to that of Theorem 2.1 already detailed, instead of Theorem 1.1, it uses Theorem 1.2.

**Corollary 3.2** If $f_j(z) \in C_S^{(k)}(\alpha)$, $(j = 1, \cdots, m)$, then

$$(f_1 \ast f_2 \ast \cdots \ast f_m)(z) \in C_S^{(k)}(\alpha).$$

**Proof.** In view of Theorem 1.2, Corollary 3.2 follows readily from Theorem 3.1 for special case when $\alpha_j = \alpha$.

**Theorem 3.3** If $F_j(z) \in TC_S^{(k)}(\alpha_j)$, $(j = 1, \cdots, m)$, then

$$(F_1 \ast F_2 \ast \cdots \ast F_m)(z) \in TC_S^{(k)}(\beta, \mu),$$

where $\beta$ and $\mu$ given by conditions in Theorem 2.1 and for $\sum_{j=1}^{m} \frac{1}{p_j} = 1$, $p_j > 1$.

**Proof.** By using the same technique as in the proof of Theorem 2.1, the required result is obtained.

**Theorem 3.4** Let the function $f_j(z) \in C_S^{(k)}(\alpha_j)$, $(j = 1, \cdots, m)$, and let $t_m(z)$ be given by (2.1). Then

$$t_m(z) \in C_S^{(k)}(\delta, \gamma),$$
where
\[
\delta = 1 - \frac{nk}{m(nk + 1)^{p-1}\left(\frac{n^{k+1-\alpha}}{1-\alpha}\right)^p - 1}, \quad \gamma = 1 - \frac{n}{m^{p-1}\left(\frac{n}{1-\alpha}\right)^p}
\]
and
\[
(k+1)^{p-2}\left(\frac{n + 1 - \alpha}{1 - \alpha}\right)^p; \quad n^{p-2}\left(\frac{n}{1-\alpha}\right)^p \geq m, \quad \alpha = \min_{1 \leq j \leq m} \alpha_j.
\]

**Proof.** Since \(f_j(z) \in C_S^{(k)}(\alpha_j)\), by using Theorem 1.2, we observe that
\[
\sum_{n=1}^{\infty} \frac{(nk + 1)(nk + 1 - \alpha_j)}{1 - \alpha_j}|a_{nk+1,j}| + \sum_{n=2}^{\infty} \frac{n^2}{1 - \alpha_j}|a_{n,j}| \leq 1, \quad (j = 1, \ldots, m).
\]

Thus the proof of Theorem 3.4 using Theorem 1.2 is precisely in the same manner as the above proof of Theorem 2.4 using Theorem 1.1.

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**References**


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