The Chaos and Control of a Prey-Predator Model with Some Unknown Parameters

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Abstract

In this paper we study the problems of chaos and control of the prey-predator model with some unknown parameters. We will use the non-linear feedback control approach to stabilize this system about its equilibrium states. The dynamic estimators of the unknown parameters will be presented. Numerical analysis and extensive illustrative examples of the uncontrolled and controlled systems were carried out by using various values of both parameters and initial densities.

Keywords: Dynamic estimators, Attractors, Unknown parameters, Asymptotic stability, Liapunov stability

1 Introduction

There is an extensive literature concerned with the dynamics of this predator prey system; for example, see [1, 2, 6, 7]. The population dynamics of the predator prey system with Holling II functional response has attracted great attention and has been studied extensively owing to its theoretical and practical significance; see [4, 5, 8]. The present paper will concern with the problem of control and chaos of prey predator system with some unknown parameters.

The results of the present paper complement and extend the recently results published by (El-Gohary and Al-Ruzaiza, 2002, 2007, 2008), see [3, 12, 20].

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This paper is organized as follows. We start in section 2 by defining the three species population which consists of two preys and one predator. The nonlinear system of differential equations governed this system is introduced. In section 3, we discuss the equilibrium states and their stability analysis of the three species prey-predator system. Further, the convergence of the subsystems to limit cycles is studied. The problem of control of the underlying system is studied in section 4. Also in this section, the Liapunov asymptotic stability of the controlled system is proved and also the necessary control inputs are obtained as nonlinear feedback. Extensive Numerical examples and numerical simulation are introduced in section 5.

2 The Mathematical Model of the system

This section is devoted to study the differential equations which governed the dynamics of the continuous time three species prey-predator populations. In this section, we will use the mathematical tools and ecological assumptions for modeling the three-species prey-predator system which consists of two competing preys and one predator.

2.1 The Basic Ecological Assumptions

We consider an eco-epidemiological system consists of three species, namely, the sound prey, the infected prey and the predator populations. We suppose that, the sound prey population grows according to a logistic law involving the whole prey population, which is best regarded as a purely descriptive equation. The transmission rate among the sound prey populations and infected prey populations follows the simple law of mass action. The disease is spread among the prey population only and that disease is not genetically inherited. The infected populations do not recover or become immune. The predator population predators mostly the infective prey and the functional response is of holling-type II.

Throughout this paper we take \( i, j = 1, 2 \) and let us define the following variables and parameters.
\begin{align*}
x_1(t) & : \text{the number of the sound prey population at time } t, \\
x_2(t) & : \text{the number of the infected prey population at time } t, \\
x_3(t) & : \text{the number of the predator population at time } t, \\
r & : \text{the intrinsic growth rate of the sound prey population,} \\
k & : \text{the ecosystem support or environmental carrying capacity,} \\
b & : \text{the rate of transmission from sound prey population to infected prey population,} \\
c & : \text{the natural death rate of infected prey,} \\
d & : \text{the death rate of predator population,} \\
\epsilon & : \text{the conversion rate,} \\
\gamma(x_2) & : \text{the predator response for the infected prey,} \\
\eta \gamma_1(x_1) & : \text{the predator response for the sound prey,}
\end{align*}

From the above assumptions we can write the following set of nonlinear differential equations:

\begin{align*}
\frac{dx_1}{dt} &= r(x_1 + x_2)\left(1 - \frac{x_1 + x_2}{k}\right) - bx_1x_2 - \eta \Phi_1(x_1, x_2)x_3, \\
\frac{dx_2}{dt} &= bx_1x_2 - \Phi_2(x_1, x_2)x_3 - cx_2, \\
\frac{dx_3}{dt} &= (\epsilon \Phi_2(x_1, x_2) + \eta \epsilon \Phi_1(x_1, x_2) - d)x_3.
\end{align*}

(1)

We will analyze the (1) with the following initial conditions:

\begin{align*}
x_1(0) > 0, \quad x_2(0) > 0, \quad x_3(0) > 0,
\end{align*}

(2)

The conditions (2) represent the conditions of positivity or biologically feasibility of the densities of sound prey, infected prey and predator populations respectively.

We assume here that the response functions \( \Phi_1 \) and \( \Phi_2 \) are increasing and bounded functions of \( x_1 \) and \( x_2 \): As an example; we shall consider these response functions as of Holling type II; given by

\begin{align*}
\Phi_1(x_1, x_2) &= \frac{m_1x_1}{a_1 + x_1}, \\
\Phi_2(x_1, x_2) &= \frac{m_2x_2}{a_2 + x_2}.
\end{align*}

(3)

Note \( m_1 \) and \( m_2 \) are the search rates and \( a_1 \) and \( a_2 \) are the search rates multiplied by handling times.

We first observe that the right-hand side of the system of equations (1) is a smooth function of the variables \((x_1, x_2, x_3)\) and the parameters, as long as these quantities are non-negative, so local existence and uniqueness properties hold in the positive octant. From the third equation of the system (1), it follows that \( x_3 = 0 \) is an invariant subset, that is, \( x_3 = 0 \) if and only if \( x_3(t) = 0 \) for
some $t$. Thus, $x_3(t) > 0$ for all $t$ if $x_3(0) > 0$: The same argument follows for
the second equation (1) if we assume $\Phi_1(0) = 0$. So, either $x_1 = 0$ in which
case the first equation of the system (1) reduces to a pure logistic law verified
by $x_1$, and $x_3$ is going to zero asymptotically; or, $x_1(t) > 0$, for all $t$. Summing
up the first two equations of system (1), we obtain

$$
\frac{d}{dt}(x_1 + x_2) = r(x_1 + x_2)(1 - \frac{x_1 + x_2}{k}) - \Phi_2(x_1, x_2)x_3 - cx_1,
$$

(4)

From (4) we see that

$$
x_1(t_0) + x_2(t_0) \leq k \Rightarrow x_1(t) + x_2(t) \leq k \forall t \geq t_0
$$

(5)

Next, we discuss the numerical solution of the system (1) for fixed values
of the system parameters and initial densities.
Figures 1 and 2 display the numerical solution and different limit cycles of the prey-predator system for the set values of the system parameters and initial densities $a_1 = 4.3, a_2 = 2.5, b = 2.1, r = 1.5, k = 1.5, r = 0.9, m_1 = 0.23, m_2 = 0.15, \epsilon = 2.5, \eta = 0.2$ and $c = 0.12, d = 0.013$ respectively, and subjected to initial state $x_1(0) = 0.25, x_2(0) = 0.45, x_3(0) = 0.95$. 
Figure 3. Three different attractors of the prey-predator system for the sets value of the system parameters and initial densities $a_1 = 5.23, a_2 = 1.25, b = 5.5, r = 4.5, k = 12.5, r = 4.5, m_1 = 0.75, m_2 = 0.95, \epsilon = 2.2, \eta = 0.15, c = 29, d = 0.19$ and subjected to initial state $x_1(0) = 0.75, x_2(0) = 0.65, x_3(0) = 0.95; a_1 = 2.95, a_2 = 0.26, b = 5.5, r = 4.5, k = 12.5, r = 4.5, m_1 = 0.75, m_2 = 0.95, \epsilon = 1.55, \eta = 0.15, c = 29, d = 0.5$ and subjected to initial state $x_1(0) = 0.45, x_2(0) = 0.65, x_3(0) = 0.95; a_1 = 2.95, a_2 = 0.2, b = 5.5, r = 4.5, k = 7.5, r = 4.5, m_1 = 0.75, m_2 = 0.95, \epsilon = 1.35, \eta = 0.15, c = 19, d = 0.85$ respectively, and subjected to initial state $x_1(0) = 0.45, x_2(0) = 0.6, x_3(0) = 0.9$ respectively.

3 Equilibrium states and Stability Analysis

This section is devoted to study the stability analysis of the continuous time three species prey-predator populations system. The equilibrium states will be determined. The chaotic behavior of the prey-predator system will be discussed.

In system (1), the number $r$ represents the growth rate of the population. The greater $r$, the faster the population reaches its carrying capacity. For $r = \infty$, one can consider that $x_1 + x_2 = k$ and system (1) reduces to the following two dimensional subsystem:

$$\frac{dx_2}{dt} = bx_2(k - x_2) - \frac{m_2x_2x_3}{a_2 + x_2} - cx_2,$$

$$\frac{dx_3}{dt} = \left(\frac{\epsilon m_2x_2}{a_2 + x_2} - d\right)x_3,$$

(6)
The reduced system (6) has the following nonnegative equilibrium states, namely:

\[ E_1 = (0, 0), \quad E_2 = (k - c/b, 0), \quad E_3 = (\bar{x}_2, \bar{x}_3), \]  

where \( \bar{x}_2 \) and \( \bar{x}_3 \) are given by

\[ \bar{x}_2 = a_2d_2/(\epsilon m_2 - \bar{d}), \quad \bar{x}_3 = \epsilon \bar{x}_2(bk - c - b\bar{x}_2)/d, \]

The first equilibrium state of the subsystem (6) has linear stability if \( bk < c \) which made the density of the infected prey is negative which is biologically not admissible so this equilibrium state is unstable. While the second equilibrium state of the subsystem (6) represents a critical case of its eigenvalues is zero, such this case needs further stability analysis.

### 4 The Control and Parameters Estimation

The adaptive control of the prey-predator system with known parameters has been studied by (El-Gohary and Al-Ruzaiza 2007). In this section we study the adaptive control of prey-predator with some unknown parameters. In order to study the adaptive control of the prey-predator system with some unknown parameters by a nonlinear feedback control approach, we start by assuming that the system (1) can be written in the following form

\[
\frac{dx_1}{dt} = \hat{r}(x_1 + x_2)\left(1 - \frac{x_1 + x_2}{k}\right) - \hat{b}x_1x_2 - \frac{\hat{\eta} \hat{m}_1 x_1 x_3}{a_1 + x_1} + u_1,
\]

\[
\frac{dx_2}{dt} = \hat{b}x_1x_2 - \frac{\hat{m}_2 x_2 x_3}{a_2 + x_2} - \hat{c}x_2 + u_2,
\]

\[
\frac{dx_3}{dt} = \left(\frac{\hat{\epsilon} \hat{m}_2 x_2}{a_2 + x_2} + \frac{\hat{\eta} \hat{m}_1 x_1}{a_1 + x_1} - \hat{d}\right)x_3 + u_3
\]

where \( u_1, u_2 \) and \( u_3 \) are external control inputs that will be suitably choice to make the trajectory of the whole system (9) that specified by the equilibrium states \( E_1(0, 0, 0) \) to be asymptotically stable about this equilibrium state of the uncontrolled system and \( \hat{r}, \hat{b}, \hat{d}, \hat{\eta}, \hat{\epsilon}, \hat{m}_j(t), (j = 1, 2) \) are the dynamic estimators of the prey-predator unknown parameters.

If \( u_i = 0, (i = 1, 2, 3) \) then the system (9) has absolutely an unstable special solution

\[ x_i(t) = 0, (i = 1, 2, 3), \quad \hat{r} = r, \quad \hat{b} = b, \quad \hat{d} = d, \quad \hat{\eta} = \eta, \quad \hat{\epsilon} = \epsilon, \quad \hat{m}_j(t) = m_j, (j = 1, 2), \]
where \( x_i = 0 \), \( i = 1, 2, 3 \) is the first one of the equilibrium states (7) of the uncontrolled system (1) that should be asymptotically stabilized using the control inputs \( u_i, \ i = 1, 2, 3 \).

Therefore, the problem is equivalent to stabilize the equilibrium state (10) with the help of the controllers \( u_i, \ i = 1, 2, 3 \).

Next, we study the adaptive control of the system (9). Adaptation is a fundamental characteristic of living organisms such as prey-predator systems since they attempt to maintain physiological equilibrium in the midst of changing environmental conditions. An approach to the design of adaptive systems is then to consider the adaptive aspects of human or animal behavior and to develop systems which behave some what analogously.

In the subsequent part of this paper we shall consider the case \( \eta = 0 \); that is to say; we assume that the predator eats only the infected prey. The case when the predator eats also a small fraction \( \eta > 0 \) of the sound prey will be briefly \( x_3 \) will be discussed nextly.

The following theorem derives the adaptive nonlinear feedback control inputs \( u_1, u_2 \) and \( u_3 \) that asymptotically stabilize the prey-predator system (1) about the equilibrium state (10).

**Theorem 3.1** Using the nonlinear feedback controllers:

\[
\begin{align*}
    u_1 & = (b x_2 - k_1)x_1 - r(x_1 + x_2)\left(1 - \frac{x_1}{k} - \frac{x_2}{k}\right), \\
    u_2 & = c x_2 + \frac{m_2 x_2 x_3}{a_2 + x_2} - (k_2 + b x_1)x_2, \\
    u_3 & = \left(d - k_3\right)x_3 - \frac{m_2 x_2 x_3}{a_2 + x_2},
\end{align*}
\]

and nonlinear updating rules of the unknown rates:

\[
\begin{align*}
    \dot{r} & = -\alpha_1(\dot{r} - r) - r(x_1 + x_2)\left(1 - \frac{x_1}{k} - \frac{x_2}{k}\right), \quad \dot{b} = -\alpha_2(\dot{r} - r), \\
    \dot{m}_2 & = -\alpha_3(\dot{m}_2 - m_2) + \frac{x_2^2 x_3}{a_2 + x_2} - \frac{x_2 x_3^2}{a_2 + x_2}, \quad \dot{c} = -\alpha_4(\dot{c} - c) + x_2^2, \\
    \dot{d} & = x_3^2 - \alpha_5(\dot{d} - d),
\end{align*}
\]

the system...
\[ \dot{x}_1 = \dot{r}(x_1 + x_2) \left( 1 - \frac{x_1 + x_2}{k} \right) - \dot{b} x_1 x_2 + u_1, \]

\[ \dot{x}_2 = \hat{b} x_1 x_2 - \frac{\hat{m}_2 x_2 x_3}{a_2 + x_2} - \hat{c} x_2 + u_2, \tag{13} \]

\[ \dot{x}_3 = \left( \frac{\hat{m}_2 x_2}{a_2 + x_2} - \hat{d} \right) x_3 + u_3 \]

will be asymptotically stable in the Liapunov sense about the equilibrium state

\[ x_i(t) = 0, \quad (i = 1, 2, 3), \quad \dot{r}(t) = r, \quad \hat{b}(t) = b, \quad \hat{c}(t) = c, \quad \hat{d}(t) = d, \quad \hat{m}_2(t) = m_2. \tag{14} \]

**Proof.** The proof of this theorem can be reached by using Liapunov stability theorem which gives sufficient conditions for asymptotic stability. Substituting (11) into (13) one can get the following nonlinear system of differential equations:

\[ \dot{x}_1 = (\dot{r} - r)(x_1 + x_2) \left( 1 - \frac{x_1}{k} - \frac{x_2}{k} \right) - (\hat{b} - b) x_1 x_2 - k_1 x_1, \]

\[ \dot{x}_2 = (\hat{b} - b) x_1 x_2 - \frac{(\hat{m}_2 - m_2) x_2 x_3}{a_2 + x_2} - (\hat{c} - c) x_2 - k_2 x_2, \tag{15} \]

\[ \dot{x}_3 = \frac{(\hat{m}_2 - m_2) x_2 x_3}{a_2 + x_2} - (\hat{d} - d) x_3 - k_3 x_3, \]

This system of nonlinear differential equations aggregated with the system (12) form a closed system of nonlinear equations which can be used to study the adaptive control of prey-predator system with some unknown parameters.

Let us consider the Liapunov function for the system which consists of the two system (12) and (15) in the form

\[ 2V = \sum_{i=1}^{3} x_i^2 + (\dot{r} - r)^2 + (\hat{b} - b)^2 + (\hat{c} - c)^2 + (\hat{d} - d)^2 + (\hat{m}_2 - m_2)^2 \tag{16} \]

Obviously, the function (16) is a positive definite function with respect to the variables \( x_1, x_2, x_3 \) and \( \dot{r}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t), \hat{m}_2(t) \) and its time derivative along the trajectory of the system consists of (12) and (15) is given by

\[ \dot{V} = -\sum_{i=1}^{3} k_i x_i^2 - \alpha_1(\dot{r} - r)^2 - \alpha_2(\hat{b} - b)^2 - \alpha_3(\hat{m}_2 - m_2)^2 - \alpha_4(\hat{c} - c)^2 - \alpha_5(\hat{d} - d)^2 \leq 0 \tag{17} \]
We can verify that the function in (17) is a negative definite function which proves the asymptotic stability of the solution (14) in the Liapunov sense. Therefore the coupled system (13) is asymptotically stable with the nonlinear feedback controllers (11), which completes the proof.

The rest of the paper is devoted to study the numerical analysis of the controlled system for different values of the system parameters and initial states.

5 Numerical Analysis and Simulation

The main objective of the numerical simulation is to explore the possibility of the chaotic behavior and the effect of the adaptive control to this behavior. Extensive numerical examples for uncontrolled three species two preys and one predator system with unknown parameters were carried out for various parameters values and different initial densities.

We start by studying the behavior the densities of the two preys and one predator with the time of the controlled system for specified values of the system parameters and initial densities. Also the estimators of the unknown parameters will discussed numerically using some values of the system parameters and initial densities.
Figures 4a, 4b and 4c display the densities of the two prey and one predator for fixed values of the system parameters and initial states.
Figures 4d, 4e, 4f, 4h, 4i, 4j and 4k display the dynamic estimators of the unknown parameters $\alpha_1, \alpha_2, \gamma_1, \gamma_2, \nu_1, \nu_2, \mu_1, \mu_2$, for the parameter values and initial densities $\alpha_1 = 2, \alpha_2 = 3, \beta_1 = 10, \beta_2 = 3, \gamma_1 = 3, \gamma_2 = 5, \nu_1 = 10, \nu_2 = 3, \mu_1 = 2, \mu_2 = 6, y_1(0) = 0.9, y_2(0) = 0.5, y_3(0) = 0.75$ and the values of control parameters are $l_1 = 2, l_2 = 5, l_3 = 7, k_1 = 5, k_2 = 4, k_3 = 3, m_1 = 2, m_2 = 6, n_1 = 10, n_2 = 3, g_1 = 4, g_2 = 5$.

Finally, we conclude that that three species prey-predator model with some unknown parameters can be asymptotically stabilized for arbitrary values of the system parameters and initial states. Further all the densities of the sound prey, infected prey and the predator are converge exponential to the equilibrium states.

6 Conclusion

This paper discussed the problem of chaos and control of three species prey-predator model with some unknown parameters. The chaotic behavior of continuous time three species prey-predator system is investigated. The unknown parameters of the model are estimated. The problem of adaptive control of three species two prey and one predator three species prey-predator model with some unknown parameters is studied. The asymptotic stability of the controlled system is proved using the Liapunov function. The necessary control inputs for this asymptotic stability is obtained as nonlinear feedback. Numerical examples are introduced.
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References


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