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A Bayesian Analysis of Order Statistics from the Generalized Rayleigh Distribution

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Abstract

The Bayes estimators for the parameter, reliability function of the generalized Rayleigh distribution are obtained. These estimators are obtained on the basis of square error and LINEX (linear-exponential) loss functions. Comparisons in terms of risks of those under squared error loss functions with Bayes estimators relative to square error loss function are given. Numerical and simulation examples are included.

Keywords: Bayes estimator; LINEX loss function; Posterior distribution; Quadratic loss function; Burr type X distribution; order statistics; Reliability function; Risk efficiency; Risk function

1. Introduction

Several authors considered different aspects of the Burr Type X and Burr type XII distributions, see for example Aludatt et al (2008), Surles and Padjett (1998), Jaheen (1995, 1996), and Ragab (2006). Recently Surles and Padjett (2001, 2005) introduced two parameter Burr type X distributions and correctly named as the generalized Rayleigh distribution.. They showed that the two parameter generalized Rayleigh distribution can be used quite effectively in modeling strength data and also modeling general lifetime data. Kundo and Raqab (2005) considered different estimators and studied how the estimator of the different unknown parameter behave for different sample sizes and for different parameter values. They compared the maximum likelihood estimators, the modified moment estimators and estimates based on percentiles by using expensive simulation techniques.

The generalized Rayleigh probability density function (pdf) is

$$f\left(x|\sigma\right)=\frac{2x\theta}{\sigma^2}e^{\frac{-x^2}{2\sigma^2}}\left(1-e^{-(\frac{x}{\sigma})^2}\right)^{\theta-1} \quad ; \quad x\geq 0,\ \ \theta\geq 0,\ \sigma>0. \tag{1.1}$$

The reliability function at mission time t , is given respectively by

$$F\left(x|\sigma\right)=\left(1-e^{-(\frac{x}{\sigma})^2}\right)^{\theta} \quad ; \quad x\geq 0,\ \ \theta\geq 0,\ \sigma>0. \tag{1.2}$$

In the estimation of reliability function use of symmetric loss function may be inappropriate as has been recognized by Canfield (1970). Overestimate of reliability function or average failure time is usually much more serious than underestimate of reliability function or mean failure time. Also, an underestimate of the failure rate results is more serious consequence than an overestimate of the failure rate. For example, in the disaster of a space shuttle (see Feynman (1987)) the management underestimated the failure rate and therefore overestimated the reliability of solid-fuel rocket booster. Varian (1975) and Zellner (1986) proposed an asymmetric loss function known as LINEX loss function which has been found to be appropriate in the situation where overestimation is more serious than underestimation or vice-versa.

When estimating a parameter θ by $\hat{\theta}$, this loss function is given by

$$L(\Delta)=e^{a\Delta}-a\Delta-1 \quad ; \quad a\neq 0, \tag{1.4}$$

and $\Delta=\frac{\hat{\theta}}{\theta}-1$. The sign and magnitude of "a" represent the direction and degree of asymmetry respectively. The positive value of "a" is used when overestimation is more serious than underestimation while negative value of "a" is used in the reverse situation. For "a" close to zero, this loss function is approximately squared error loss and therefore almost symmetric. Several authors including Basu and Ebrahimi (1991), Rojo (1996), Soliman (2000) and Zellner (1986) have used this loss function in various estimation and prediction problems.

If we define $\Delta_1=\hat{\theta}-\theta$, then $L(\Delta_1)$ is equivalent to the loss function used by Varian (1975) and Zellner (1986). And if we define $\Delta_2=\left(\frac{\hat{\theta}}{\theta}\right)^2-1$, then $L(\Delta_2)$ is equivalent to the loss function used by Soliman (2000).

For situations where life tester has no prior information about the parameter θ, we may use the quasi-density prior in the following form:

$$p(\theta)\propto\frac{1}{\theta^d}\ , \qquad \theta>0, \mathrm{d}>0.$$

If $d=1$ we get a non-informative prior (Jeffrey's (1961))

$$p_1(\theta) \propto \frac{1}{\theta} , \qquad \theta > 0, \tag{1.5}$$

Also, if $d = 3$ we get the asymptotically invariant prior, proposed by Hartigan (1964)

$$p_2(\theta) \propto \frac{1}{\theta^3} , \qquad \theta > 0, \tag{1.6}$$

The plan of the article is as follows: In sections 2, Bayes estimates for the shape parameter θ of the generalized Rayleigh distribution are obtained. The estimates are given on based the square error and LINEX loss function $L(\Delta_2)$ by using $p_i(\theta),(i=1,2)$ as prior distributions, the risk of estimates have been obtained. Comparison in terms of risk with the estimates of θ under squared error loss and LINEX loss when the prior distributions are $p_i(\theta),(i=1,2)$ have been made. Also, we give a numerical example to compare our results. In section 3 we obtain Bayes estimates of $R(t)$ under prior distributions $p_i(\theta),(i=1,2)$. and compared with those corresponding to Maximum likelihood estimate.

2. Bayes Estimate of θ.

In this section, we shall be concerned with estimation of the shape parameter θ of the generalized Rayleigh model. Suppose we observe n ordered values $X_{(1)}, X_{(2)},..., X_{(n)}$ from the generalized Rayleigh distribution pdf given by (1.1). The likelihood function (LF) is given by

$$f(\underline{x}\,\theta) = \prod_{i=1}^{n} f(x_i) \tag{2.1}$$

where $\underline{x} = (x_1, x_2,..., x_n)$ and $H(.)$ is the hazard function corresponding to pdf $f(.)$. It follows, from (1.1),(1.2) and (1.2), that

$$f(\underline{x}|\theta) = \left(\frac{2\theta}{\sigma^2}\right)^n \prod_{i=1}^{n} \left(1 - e^{(\frac{x_i}{\sigma})^2}\right)^{\theta\text{-}1} \quad ; \quad \theta > 0 \tag{2.2}$$

2.1 Maximum likelihood Estimation of θ .

The natural logarithm of the LF (2.2) is

$$\ell = \ln f(\underline{x}|\sigma) \propto \; n \ln\theta + \sum_{i=1}^{n} (\theta - 1)\ln\left(1 - e^{-(\frac{X_i}{\sigma})^2}\right) \tag{2.3}$$

Assuming that the scale parameter σ is known, without loss of generality, we can assume that σ =1. The maximum likelihood estimate (MLE) of the parameter θ can be shown to be

$$\hat{\theta}_{ML} = \frac{n}{T} \ , \tag{2.4}$$

where

$$T = -\sum \ln(1 - e^{-x^2}) . \tag{2.5}$$

Note that, if $X_i's$ are independent and identically distributed one-parameter generalized Rayleigh distribution as in (1.1), with $\sigma = 1$, then it is easy to see that $-\theta \sum_{i=1}^{n} \ln(1 - e^{-x^2})$ has gamma distribution with shape parameter n and scale parameter 1, giving the probability density function of T as

$$h(t) = \frac{\theta^n}{\Gamma(n)} t^{n-1} e^{-\theta t} \quad , \qquad t > 0 \ . \tag{2.6}$$

2.2 Bayes Estimator of θ based on squared error loss function.

Case (i): We assume that the prior density $p_1(\theta)$, combining the likelihood function (2.2) and the prior $p_1(\theta)$, the posterior density of θ is

$$\pi_1(\theta|\underline{x}) = \frac{\theta^{n-1} T^n e^{-\theta}}{\Gamma(n)} \quad , \qquad \theta > 0 \tag{2.7}$$

By using (2.6) under square error loss ($L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$), the Bayes estimate of θ denoted by $\hat{\theta}_{BS1}$ is the posterior mean $E_{\pi_1}(\theta)$, is

$$\hat{\theta}_{BS1} = \frac{n}{T} \ , \quad t > 0 \tag{2.8}$$

where T is given in (2.5).

Case (ii): We assume that the prior $p_2(\theta)$ in (1.6), combining the likelihood function (2.2) and the prior $p_2(\theta)$, the posterior density of θ is given by

$$\pi_2(\theta|\underline{x}) = \frac{\theta^{n-3} T^{n-2} e^{-\theta T}}{\Gamma(n-2)} \quad , \qquad \theta > 0 \tag{2.9}$$

The Bayes estimate of θ denoted by $\hat{\theta}_{BS2}$ is the posterior mean $E_{\pi_2}(\theta)$, hence

$$\hat{\theta}_{BS2} = \frac{n-2}{T}, \qquad t > 0 \tag{2.10}$$

2.3 Bayes Estimator of θ based on LINEX loss function.

Under the LINEX loss function (1.4), the posterior expectation of the loss function $L(\Delta_2)$ with respect to $\pi_i(\theta|\underline{x})$ in (2.6) and (2.8), is

$$E_{\pi_i}[L(\Delta_2)] = \int_0^\infty \left\{ e^{a\left[\hat{\theta}^2 - \theta\right]} - a\left[\hat{\theta} - \theta\right] - 1 \right\} \pi_i(\theta|\underline{x})\, d\theta \quad , i = 1,2, \tag{2.11}$$

i.e.

$$E_{\pi_i}[L(\Delta_2)] = e^{-a} E_{\pi_i}\left[\exp\left(a(\hat{\theta} - \theta)\right)\right] - aE_{\pi_i}\left[\left(a(\hat{\theta} - \theta)\right)\right] \text{-}1 \quad , i = 1,2. \tag{2.12}$$

The value of $\hat{\theta}$ that minimizes the posterior expectation of the loss function $L(\Delta_2)$, denoted by $\hat{\theta}_{BLi}$, $(i = 1,2)$; are obtained by solving the equations:

$$\frac{dE_{\pi_i}[L(\Delta_2)]}{d\hat{\theta}} = E_{\pi_i}\left[e^{-a} \exp\left(a(\hat{\theta} - \theta)\right)\right] - E_{\pi_i}\left(\hat{\theta}\right) = 0 \quad , i = 1,2, \tag{2.13}$$

that is, $\hat{\theta}_{BLi}$ is the solution to the following equations:

$$E_{\pi_i}\left[ae^{-a\theta} e^{a\hat{\theta}}\right] - a = \qquad , i = 1,2, \tag{2.14}$$

provided that all expectations exist and are finite. Using (2.6) and (2.8) in (2.13) respectively when $i = 1,2,$ we get the optimal estimate of θ relative to $L(\Delta_2)$ is:

$$\hat{\theta}_{BL1} = \frac{n}{a} \ln \frac{a+T}{T}, \tag{2.15}$$

and

$$\hat{\theta}_{BL2} = \frac{n-2}{T} \ln \frac{a+T}{T}. \tag{2.16}$$

2.4.2 The risk efficiency of estimators $\hat{\theta}_{Bli}$ and $\hat{\theta}_{Bsi}$ $(i = 1,2)$ under squared error loss $\mathbf{L(\Delta_1)}$.

The risk functions (the expected loss $E_X[L(\Delta)]$) are denoted by $R_{S_i}(\hat{\theta}_{BLi})$ and $R_{S_i}(\hat{\theta}_{BSi})$, $i = 1,2$, with subscript S_i denoting squared error loss and are given by (2.23)

Thus
$$R_{S_1}(\hat{\theta}_{BLi}) = \int_0^\infty \left\{ \frac{n}{a} \ln\left(\frac{a+t}{t}\right) - \frac{2n\theta}{t} \ln\left(\frac{a+t}{t}\right) + \theta^2 \right\} \frac{\theta^n}{\Gamma(n)} t^{n-1} e^{-\theta t} dt, \tag{2.24}$$

$$R_{S_1}(\hat{\theta}_{BLi}) = \int_0^\infty \left\{ \frac{n-2}{a} \ln\left(\frac{a+t}{t}\right) - \frac{2n\theta}{t} \ln\left(\frac{a+t}{t}\right) + \theta^2 \right\} \frac{\theta^n}{\Gamma(n)} t^{n-1} e^{-\theta t} dt. \tag{2.26}$$

Also, the risk of θ_{BSi} under squared error loss are

$$R_{S_i}(\hat{\theta}_{BSi}) = \int_0^\infty \left(\hat{\theta}_{BSi} - \theta\right)^2 h(t)dt \quad , i = 1,2. \tag{2.27}$$

Thus

$$R_{S_1}(\hat{\theta}_{BS1}) = \int_0^\infty \left[(\frac{n}{t})^2 - \frac{2n\theta}{t} + \theta^2 \right] \frac{\theta^n}{\Gamma(n)} t^{n-1} e^{-\theta t} dt, \tag{2.28}$$

i.e.

$$R_{S_1}(\hat{\theta}_{BS1}) = \frac{(n+2)\theta^2}{(n-1)(n-2)} \quad , \tag{2.29}$$

and

$$R_{S_1}(\hat{\theta}_{BS1}) = \int_0^\infty \left[(\frac{n-2}{t})^2 - \frac{2(n-2)\theta}{t} + \theta^2 \right] \frac{\theta^n}{\Gamma(n)} t^{n-1} e^{-\theta t} dt, \tag{2.30}$$

i.e.

$$R_{S_2}(\hat{\theta}_{BS2}) = \frac{\theta^2}{n-1} \quad ,$$

The efficiency of $\hat{\theta}_{BLi}$ with respect to $\hat{\theta}_{BSi}$ under squared error loss is defined

$$RE_{S_i}(\hat{\theta}_{BLi}, \hat{\theta}_{BSi}) = \frac{R_{S_i}(\hat{\theta}_{BSi})}{R_{S_i}(\hat{\theta}_{BLi})} \quad , i = 1,2. \tag{2.31}$$

It may be seen from expressions of risks of the estimators that an analytical comparison of these risks may not be possible, therefore, we decide for an empirical comparison.

2.5 Numerical example.

To compare the proposed estimator $\hat{\theta}_{BLi}(i = 1,2)$ with the usual estimators $\hat{\theta}_{BSi}(i = 1,2)$, the risk function are computed so as to see how $\hat{\theta}_{BLi}$ performs as compared to $\hat{\theta}_{BSi}$ when true loss is squared error. A comparison of this type may be needed to check whether an estimator is inadismissible under some loss function. If it is so the estimator would not be used for the losses specified by that loss function. For this purpose the risks of the estimators and risk efficiency have been computed.

Example 2.1: A sample of size 20 is generated from the generalized Rayleigh model (Burr type X) given by (1.1) with $\theta = 1$ and written in order form as:

0.2189, 0.3803, 0.4032, 0.4735, 0.4776, 0.5087, 0.5280, 0.5943, 0.6160, 0.6634, 0.8731, 0.8887, 0.9208, 1.1863, 1.2025, 1.4684, 1.5314, 1.5772, 1.6416 and 1.8764

we compute the estimates and their corresponding risk efficiency for θ, in table I and table II as follows:

Table I

The estimators $\hat{\theta}_{BL1}, \hat{\theta}_{BS1}$ the risk efficiencies $RE_{S_1}(\hat{\theta}_{BL1}, \hat{\theta}_{BS1})$ under the prior $p_1(\theta)$ for the variations in a sample $n = 20$.

$\hat{\theta}_{BS1} = 1.1025$, $R_{S1}(\hat{\theta}_{BS1}) = 0.0584$			
a	$\hat{\theta}_{BL1}$	$Rs_1(\hat{\theta}_{BL1})$	$RE_{s1}(\hat{\theta}_{BL1}, \theta_{BS1})$
2	0.9908	8.1079	0.0072
4	0.9461	3.2867	0.0017
6	0.9063	1.7725	0.032
8	0.8707	1.0709	0.054
10	0.8234	0.6859	0.085

Table II

The estimators $\hat{\theta}_{BL2}, \hat{\theta}_{BS2}$ the risk efficiencies $RE_{S_2}(\hat{\theta}_{BL2}, \hat{\theta}_{BS2})$ under the prior $p_2(\theta)$ for the variations in a sample $n = 20$

$\hat{\theta}_{BS2} = 0.93923$, $R_{S2}(\hat{\theta}_{BS2}) = 0.05263$			
a	$\hat{\theta}_{BL2}$	$RS_2(\hat{\theta}_{BL2})$	$RE_{s2}(\hat{\theta}_{BL2}, \theta_{BS2})$
2	0.8917	7.2971	0.0072
4	0.8515	2.9850	0.018
6	0.8157	1.5953	0.033
8	0.7836	0.9638	0.055
10	0.7546	0.6173	0.085

We compute and report the estimates and their corresponding risk efficiencies for θ. Table I and II show Bayes LINEX estimators $\hat{\theta}_{BLi}(i = 1,2)$, risk efficiency, ($i$= 1,2) of $RE_{S_i}(\hat{\theta}_{BLi}, \hat{\theta}_{BSi})$ under squared error loss. From table I and II (estimators based on prior density functions, $p_i(\theta), (i = 1,2)$) we note:

The risk efficiency $RE_{s_i}(\hat{\theta}_{BLi}, \hat{\theta}_{BSi})$, ($i$= 1,2) is smaller than one for all value of "a" ($a = 2,\ 4,\ 6,\ 8,\ 10$) which indicates that the propose estimators $\hat{\theta}_{BLi}(i = 1,2)$ is preferable to $\hat{\theta}_{BSi}(i = 1,2)$.

3. Bayes estimators of the reliability function.

Let $R = \overline{F}(x)$ be the probability that a system will survive a specified mission time t. Then , the reliability function of Burr type X distribution is obtained as the following

$$R(x) = 1 - (1 - e^{-x^2})^{\theta}, \tag{3.1}$$

and the maximum likelihood estimate of R is given by

$$\hat{R}(x) = 1 - \left(1 - e^{-x^2}\right)^{\hat{\theta}_{Ml}}, \quad x > 0\,;\ \theta > 0. \tag{3.2}$$

Where $\hat{\theta}_{Ml}$ is given in (2.5).

By substituting $\theta = \dfrac{\ln(1-R)}{\ln(1-e^{-x^2})}$ in (2.6), we obtain the posterior distribution of R under $p_1(\theta)$ in (1.5) as the following:

$$\pi(R|x) = \frac{t^n}{\Gamma(n)}\left(\frac{\ln(1-R)}{\ln(1-e^{-x^2})}\right)^n \frac{(1-R)}{(1-R)\ln(1-e^{-x^2})}\exp\left(-\frac{t}{\ln(1-e^{-x^2})}\right), \tag{3.3}$$

where t is given in (2.5).

The Bayes estimate of R under square error loss function s given by the following

$$\hat{R}_B = E(R) = \int_0^1 [1-(1-R)]\,\pi(R|x)\,dR$$

$$= 1 - \frac{t^n}{\Gamma(n)[\ln(1-e^{-x^2})]} \int_0^1 [\ln(1-R)]\,Ln(1\ R)]^{n-1}(1-R)\exp\left(\frac{-t}{\ln(1-e^{-x^2})}\right)dR. \tag{3.4}$$

By using the transformation $1 - R = e^z$ in (3.4) , we get

$$\hat{R}_B = 1 + \left[\frac{t}{t - \ln(1-e^{-x^2})}\right]^n. \tag{3.5}$$

Similar result obtained under $p_2(\theta)$ given in (1.6)).

Example 3.1

One samples does not tell us much. We generate 1000 samples of sizes n= 5(5)30 from (1.1) with value of θ=5(5)30. We compute $\hat{\theta}_{Ml}$ and the Bayes estimate $\hat{R}_B$ based on the prior density function (1.5). The results with corresponding values of θ and n are given in Table 3. It can be seen that the maximum likelihood estimate are more appropriate than Byes estimators $\hat{R}_B$ of reliability function.

Table 3

Estimates of maximum likelihood and Bayes estimate

of reliability function of generalized Rayleigh distribution

n	θ	True value of R	$\hat{R}_{ML}$	R_B
5	5	.899	.756	1.151
	10	.989	.999	1.044
	15	.998	.998	1.017
	20	.999	1.000	1.008
	25	.999	.999	1.004
	30	.999	1.000	1.002
10	5	.899	.967	1.127
	10	.989	.995	1.027
	15	.998	.999	1.007
	20	.999	1.000	1.002
	25	.999	.999	1.001
	30	.999	1.000	1.000
15	5	.899	.969	1.118
	10	.989	.975	1.021
	15	.998	.999	1.005
	20	.999	.999	1.001
	25	.999	.999	1.000
	30	.999	1.000	1.000
20	5	.899	.784	1.114
	10	.989	.987	1.018
	15	.998	.999	1.003
	20	.999	.999	1.000
	25	.999	.999	1.000
	30	.999	1.000	1.000
25	5	.899	.869	1.112
	10	.989	.988	1.016
	15	.998	.998	1.003
	20	.999	.999	1.000
	25	.999	1.000	1.000
	30	.999	1.000	1.000
30	5	. 899	.840	1.110
	10	.989	.997	1.015
	15	.998	.995	1.002
	20	.999	.999	1.000
	25	.999	.999	1.000
	30	.999	.999	1.000

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