Note on Buffon’s Problem

with a Long Needle

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Abstract. In the short note, we discuss the Buffon needle problem and obtain the probability of that the needle intersect the parallel lines in the case \( l > d \) by using two different ideas.

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1. Introduction

In the classical formulation of the Buffon needle problem [2] a needle of length \( l \) is thrown at random onto a plane ruled by parallel lines distance \( d \) apart \( l < d \), and one asks for the probability of an intersection. In case \( l > d \) there can be several intersections. The purpose of this note is to discuss the probability by using two different ideas.

The probability of the number of crossings is as given by Kendall and Moran [3]. Diaconis [1] obtained the distribution of the number of intersections and approximate moments for large \( l \) are derived, and showed that the distribution converges weakly to an arc-sine law as \( l/d \to \infty \). In [4], Morton considered a
plane containing a set of curves. Suppose a separate set of curves is constrained to lie at random on a particular region of this plane. In his note an expression is obtained for the expected number of intersections between the two sets of curves, and it is shown that the angles between the two sets at their points of intersection are distributed as $\frac{1}{2} \sin \theta$.

2. Main results

In this section, we will show different ideas to obtain the probability that the needle intersect the parallel lines in the case $l > d$. For the convenience of readers, let $A$ denote the event that the needle intersect the parallel lines, and for any set $B$ on plane, let $S_B$ denote the area of $B$.

2.1. One idea. Without loss the generality, let $x$ denote the distance of the midpoint of the needle from the nearest parallel line and let $\phi$ denote the angle between the needle and this parallel line, then it is easy to see

$$0 \leq x \leq \frac{d}{2}, \quad 0 \leq \phi \leq \pi.$$ 

So we can consider the sample space

$$\Omega = \left\{(x, \phi) \mid 0 \leq x \leq \frac{d}{2}, \quad 0 \leq \phi \leq \pi \right\}$$

and the area of $\Omega$ is $S_\Omega = \pi d/2$. Hence, the problem is equivalent to throwing the point $(x, \phi)$ on a plane which yields

$$A = \left\{(x, \phi) \mid 0 \leq x \leq \frac{l}{2} \sin \phi, \quad 0 \leq \phi \leq \pi \right\}.$$ 

Since

$$S_A = 2 \int_0^{\arcsin \frac{d}{l}} \frac{l}{2} \sin \phi d\phi + \int_{\arcsin \frac{d}{l}}^{\pi - \arcsin \frac{d}{l}} \frac{d}{2} d\phi,$$

then

$$P(A) = \frac{S_A}{S_\Omega} = \frac{2(l - \sqrt{l^2 - d^2}) + d(\pi - 2 \arcsin \frac{d}{l})}{\pi d}.$$
2.2. The other idea. Let $\varphi$ denote the angle between the needle and the parallel line and

$$\theta = \arcsin \frac{d}{l}.$$ 

Then if $\theta \leq \varphi \leq \pi - \theta$, then the needle intersect affirmatively the parallel lines and the probability of the needle intersecting the parallel lines is 1.

If $0 \leq \varphi < \theta$ or $\pi - \theta < \varphi \leq \pi$, then

$$A \iff \frac{l}{2} \sin \varphi > \frac{d}{2}.$$ 

Let the event $\{\theta \leq \varphi \leq \pi - \theta\}$ be denoted by $A_1$ and the event $\{0 \leq \varphi < \theta\} \cup \{\pi - \theta < \varphi \leq \pi\}$ be denoted by $A_2$, then

$$P(A) = P(A_1)P(A|A_1) + P(A_2)P(A|A_2)$$

$$= \frac{\pi - 2\theta}{\pi} + \frac{2\theta}{\pi} \int_{\theta}^{\frac{\pi}{2}} \frac{l}{2} \sin \varphi \, d\varphi$$

$$= \frac{\pi - 2\theta}{\pi} + \frac{2(l - \sqrt{l^2 - d^2})}{\pi d}$$

$$= 2(l - \sqrt{l^2 - d^2}) + d(\pi - 2 \arcsin \frac{d}{l}).$$

(2.2)

2.3. Some remarks. Recall that if $l < d$, then

$$P(A) = \frac{2l}{\pi d}.$$ 

Comparing (2.2) with above probability and as $l \to d$, then

$$\frac{2(l - \sqrt{l^2 - d^2}) + d(\pi - 2 \arcsin \frac{d}{l})}{\pi d} \to \frac{2l}{\pi d}.$$ 

For the case $l > d$, we can also estimate $\pi$ by using (2.2) and the fact that $P(A)$ is approximately equal to the frequency of the event $A$ appearing as the numbers of throwing the needle is enough large.

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References


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