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Restarted Adomian Decomposition Method to Systems of Nonlinear Algebraic Equations

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Abstract

In this paper we apply restarted Adomian decomposition method, based on standard Adomian method for solving system of nonlinear algebraic equations. Illustrative examples have been presented, to demonstrate the method and the obtained results are compared with results of the standard Adomian decomposition method.

Keywords: Adomian decomposition method; Restarted Adomian method; System of nonlinear algebraic equations

1 Introduction

Decomposition method was first introduced by Adomian since the beginning of the 1980's for solving wide range of problems whose mathematical models yield equation or system of equation involving algebraic, differential, integral and integro-diffrential [1, 2, 3]. This iterative method has been proven to be rather successful in dealing with linear problems as well as nonlinear.

Adomian gives the solution as an infinite series usually converging to an accurate solution. Abbaui and Cherruault [4] applied the standard Adomian decomposition on simple iteration method to solve $f(x) = 0$, where $f(x)$ is a nonlinear function and proved the convergence of its series solution. E.

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Babolian et al. [5] applied the standard Adomian's method to solved a system of nonlinear equations.

Restarted Adomian method, based on standard Adomian method was introduced by E. Babolian et al. [6] for algebraic equations. It is the purpose of this paper to extend restarted Adomian method for solving system of nonlinear algebraic equations. We will show by some examples that the convergence rate of the solution series will be accelerated using Adomian's restarted method.

This paper has been organized as follow. Section 2 deals with the analysis of the standard ADM for a system of nonlinear equations. In section 3, we introduce restarted ADM for systems of nonlinear algebraic equations. The restarted ADM and the standard ADM with illustrative examples have been compared in section 4. The conclusions are summarized in section 5.

2 The Principle of ADM for system of nonlinear equations

Consider the following system of nonlinear equations

$$f_i(x_1, \ldots, x_n) = 0, \quad i = 1, 2, \ldots, n \tag{1}$$

where $f_i : \Re^n \to \Re$. Equation (1) can be written in the canonical form

$$x_i = c_{i,0} + N_{i,0}(x_1, \ldots, x_n), \quad i = 1, 2, \ldots, n, \tag{2}$$

where $c_{i,0}$'s are constants and $N_{i,0}$'s are nonlinear functions of their arguments generally. The standard ADM [5] uses the solution x_i in terms of the series

$$x_i = \sum_{j=0}^{\infty} x_{i,j} \quad i = 1, 2, \ldots, n, \tag{3}$$

and the nonlinear functions N_{i0}'s are expressed in terms of an infinite series called Adomian polynomials as

$$N_{i,0}(x_1, \ldots, x_n) = \sum_{j=0}^{\infty} A_{i,j} \quad i = 1, 2, \ldots, n, \tag{4}$$

where $A_{i,j}$'s depend upon $x_{1,0}, x_{1,1}, \ldots, x_{1,j}, x_{2,0}, x_{2,1}, \ldots, x_{2,j}, \ldots, x_{n,1}, \ldots, x_{n,j}$. In view of the equations (3) and (4)

$$N_{i,0}(\sum_{j=0}^{\infty} x_{1,j}\lambda^j, \ldots, \sum_{j=0}^{\infty} x_{n,j}\lambda^j) = \sum_{j=0}^{\infty} A_{i,j}, \quad i = 1, 2, \ldots, n, \tag{5}$$

which yields

$$A_{ij} = \frac{1}{j!}[\frac{d^j}{d\lambda^j} N_{i,0}(\sum_{j=0}^{\infty} x_{1,j}\lambda^j, \ldots, \sum_{j=0}^{\infty} x_{n,j}\lambda^j)]_{\lambda=0}, \quad i = 1, 2, \ldots, n, \tag{6}$$

where λ is the parameter introduced for convenience. Hence equation (2) can be written as

$$\sum_{j=0}^{\infty} x_{i,j} = c_{i,0} + \sum_{j=0}^{\infty} A_{i,j}, \quad i = 1, 2, \ldots, n. \tag{7}$$

The ADM defines the components $x_{i,j}, j \geq 0$, by the following recursive relation

$$x_{i,0} = c_{i,0}, \tag{8}$$

$$x_{i,j+1} = A_{i,j}, \quad i = 1, 2, \ldots, n, \qquad j = 0, 1, \ldots. \tag{9}$$

Finally the solution x_i can be approximated by the truncated series

$$\varphi_{i,k} = \sum_{j=0}^{k-1} x_{i,j} \tag{10}$$

that

$$\lim_{k \to \infty} \varphi_{i,k} = x_i, \quad i = 1, 2, \ldots, n. \tag{11}$$

In computing x_i, choosing large values for k, increases the number of terms in the expression of $A_{i,n}$ and this causes propagation of round off errors. On the other hand, the factor $1/n!$ in the formula of $A_{i,n}$ makes it very small. Considering these points, we introduce a new algorithm based on Adomian method to improve the accuracy dramatically.

3 Restarted ADM for the system (1)

In this section, we extend the restart Adomian decomposition method [6] for the system (1). In the new algorithm, we rewrite the equation (2) so that $c_{i,0}$ be a better starting point. Toward this end, suppose that we have the system of nonlinear equation (1) with the exact solution $X^* = (x_1^*, \ldots, x_n^*)$.

Let $c_{i,1}$ be a point close to x_i^* such that

$$|x_i^* - c_{i,1}| < |x_i^* - c_{i,0}|, \quad i = 1, 2, \ldots, n. \tag{12}$$

Now add $c_{i,1} - c_{i,0}$ to both sides of (2)

$$x_i - N_{i,1} = c_{i,1}, \quad i = 1, 2, \ldots, n, \tag{13}$$

where

$$N_{i,1} = N_{i,0} - (c_{i,1} - c_{i,0}), \quad i = 1, 2, \ldots, n. \tag{14}$$

It is clear that X^* is the exact solution of (2). Now we can solve the equation (2) with the Adomian method instead of the equation(1). Therefore we suggest the following algorithm "the restarted Adomian method" for solving

$$x_i - N_{i,0} = c_{i,0}, \quad i = 1, 2, \ldots, n. \tag{15}$$

3.1 Restarted Adomian algorithm

Choose small positive number ε and small natural number m.

Step(1)

$$For \quad j = 0, 1, 2, \ldots \quad do$$

$$x_{i,0} = c_{i,0},$$

$$x_{i,1} = A_{i,0},$$

$$\vdots$$

$$x_{i,m} = A_{i,m-1},$$

Step(2)

$$x_i^{(j+1)} = c_{i,j+1} := x_{i,0} + \ldots + x_{i,m},$$

$$if \quad |c_{i,j+1} - c_{i,j}| < \varepsilon \quad stop$$

Step(3)

$$N_{i,j+1} = N_{i,0} - (c_{i,j+1} - c_{i,0})$$

$$end(for)$$

The number of terms in the expression for A_n increases by m and this causes propagation of round of errors, therefore we choose small values for m, say $2 \leq m \leq 5$.

4 Numerical examples

In this section we apply the restarted Adomian's decomposition method (RADM) introduced in last section to solve the examples of the systems of algebraic equations. Two examples are considered and the results compared with the standard ADM. Mathematica 5 is used to carry computations.

Example 1: Consider the system of nonlinear algebraic equations

$$\begin{cases} x_1^2 - 10x_1 + x_2^2 + 8 = 0 \\ x_1 x_2^2 + x_1 - 10x_2 + 8 = 0 \end{cases} . \tag{16}$$

with the exact solution $X^* = (x_1^*, x_2^*)^t = (1,1)^t$. Rewriting (16) in the canonical form

$$\begin{cases} x_1 = 0.8 + 0.1x_1^2 + 0.1x_2^2 \\ x_2 = 0.8 + 0.1x_1x_2^2 + 0.1x_1 \end{cases} . \tag{17}$$

and using (3), we have

$$\begin{cases} \sum_{j=0}^{\infty} x_{1,j} = 0.8 + 0.1 \sum_{j=0}^{\infty} A_{1,j}(x_1^2) + 0.1 \sum_{j=0}^{\infty} A_{1,j}(x_2^2) \\ \sum_{j=0}^{\infty} x_{2,j} = 0.8 + 0.1 \sum_{j=0}^{\infty} A_{2,j}(x_1x_2^2) + 0.1 \sum_{j=0}^{\infty} A_{2,j}(x_1) \end{cases} \tag{18}$$

Applying Adomian decomposition method introduced in section 2 and calculating seven terms of the solution series we have

$$x_1 \simeq \varphi_{1,7} = x_{1,0} + x_{1,1} + x_{1,2} + \ldots + x_{1,6} = 0.997853$$

$$x_2 \simeq \varphi_{2,7} = x_{2,0} + x_{2,1} + x_{2,2} + \ldots + x_{2,6} = 0.997562$$

the absolute errors relative to the exact solutions is

$$AE_1 = |x_1^* - \varphi_{1,7}| = 2.14 \times 10^{-3}$$

$$AE_2 = |x_2^* - \varphi_{2,7}| = 2.43 \times 10^{-3}$$

Now applying restarted Adomian decomposition by calculate seven terms, but in two steps. The results are

$$x_1^{(1)} = x_{1,0} + x_{1,1} + x_{1,2} + x_{1,3} = 0.985513$$

$$x_1^{(2)} = x_1^{(1)} + x_{1,4} + x_{1,5} + x_{1,6} = 0.999037$$

$$x_2^{(1)} = x_{2,0} + x_{2,1} + x_{2,2} + x_{2,3} = 0.984688$$

$$x_2^{(2)} = x_2^{(1)} + x_{2,4} + x_{2,5} + x_{2,6} = 0.999029$$

$$RE_1 = |x_1^* - x_1^{(2)}| = 9.63 \times 10^{-4}$$

$$RE_2 = |x_2^* - x_2^2| = 9.70 \times 10^{-4}$$

Comparing the absolute errors of two methods shows that RADM in two steps gives more accurate results than the standard ADM.

Example 2: Consider the system of nonlinear algebraic equations

$$\begin{cases} x_1^3 + x_2^3 - 6x_1 + 3 = 0 \\ x_1^3 - x_2^3 - 6x_2 + 2 = 0 \end{cases} . \tag{19}$$

with the exact solution $X^* = (x_1^*, x_2^*)^t = (0.53236428, 0.35125372)^t$. The system can be written in the canonical form

$$\begin{cases} x_1 = \dfrac{1}{2} + \dfrac{1}{6}x_1^3 + \dfrac{1}{6}x_2^3 \\ x_2 = \dfrac{1}{3} + \dfrac{1}{6}x_1^3 - \dfrac{1}{6}x_2^3 \end{cases} . \tag{20}$$

Using (3) we have

$$\begin{cases} \sum_{j=0}^{\infty} x_{1,j} = \dfrac{1}{2} + \dfrac{1}{6}\sum_{j=0}^{\infty} A_{1,j}(x_1^3) + \dfrac{1}{6}\sum_{j=0}^{\infty} A_{1,j}(x_2^3) \\ \sum_{j=0}^{\infty} x_{2,j} = \dfrac{1}{3} + \dfrac{1}{6}\sum_{j=0}^{\infty} A_{2,j}(x_1^3) - \dfrac{1}{6}\sum_{j=0}^{\infty} A_{2,j}(x_2^3) \end{cases} \tag{21}$$

Calculate five terms of series we obtain the approximates

$$x_1 \simeq \varphi_{1,5} = x_{1,0} + x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 0.53229341$$

$$x_2 \simeq \varphi_{2,5} = x_{2,0} + x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = 0.35121114$$

$$AE_1 = |x_1^* - \varphi_{1,5}| = 7.08 \times 10^{-5}$$

$$AE_2 = |x_2^* - \varphi_{2,7}| = 4.25 \times 10^{-5}$$

Applying restarted ADM, we have

$$x_1^{(1)} = x_{1,0} + x_{1,1} + x_{1,2} = 0.53119642$$

$$x_1^{(2)} = x_1^{(1)} + x_{1,3} + x_{1,4} = 0.53233286$$

$$x_2^{(1)} = x_{2,0} + x_{2,1} + x_{2,2} = 0.35055513$$

$$x_2^{(2)} = x_2^{(1)} + x_{2,3} + x_{2,4} = 0.35123519$$

$$RE_1 = |x_1^* - x_1^{(2)}| = 3.14 \times 10^{-4}$$

$$RE_2 = |x_2^* - x_2^2| = 1.85 \times 10^{-4}$$

In which the errors are calculated relative to the exact solutions. Comparing the errors of results of two methods (RADM and ADM), shows that RADM gives more accurate results than the standard ADM.

5 Conclusion

In this paper, we applied the RADM and ADM to approximate the solutions the system of nonlinear algebraic equations. The numerical results show that the RADM gives more accurate approximate solutions than ADM.

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