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On New Subclass of Analytic Univalent Functions for Operator on Hilbert Space

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Abstract

In the present paper, the authors introduce and study new subclass of normalized analytic univalent function $S_w^*(k,\beta)$ for some operator on Hilbert space.

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1 Introduction

Let $\mathcal{A}$ be the class of functions f *normalized* by

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n, \tag{1}$$

which are *analytic* in the open unit disk $D = \{z \in C : |z| < 1\}$.

As usual, we denote by S the subclass of $\mathcal{A}$, consisting of functions which are also *univalent* in D. We recall here the definitions of the well-known classes of starlike function and convex functions:

$$S^* = \left\{ f \in A : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in D \right\},$$

$$S^c = \left\{ f \in A : \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in D \right\},$$

Let w be a fixed point in D and $A(w) = \{f \in H(D) : f(w) = f'(w) - 1 = 0\}$. In [23], Kanas and Ronning introduced the following classes

$$S_w = \{f \in A(w) : f\, is\, univalent\, in\, D\}$$

$$ST_w = \left\{ f \in A(w) : \text{Re}\left(\frac{(z-w)\, f'(z)}{f(z)}\right) > 0, z \in D \right\}, \tag{2}$$

$$CV_w = \left\{ f \in A : 1 + \left(\text{Re}\frac{(z-w)\, f''(z)}{f'(z)}\right) > 0, z \in D \right\}. \tag{3}$$

Later Acu and Owa [15] studied the classes extensively. The class S_w^* is defined by geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class S_w^c is defined by the property that the image of any circular arc centered at w is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [1] and [2] for uniformly starlike and convex functions, except that in this case the point w is fixed.

Let S_w denoted the subclass of $A(w)$ consisting of the function of the form

$$f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n \qquad (a_n \geq 0) \tag{4}$$

The function f in S_w is said to be starlike functions of order β if and only if

$$\text{Re}\left(-\frac{(z-w)f'(z)}{f(z)}\right) > \beta, \qquad ((z-w) \in U) \tag{5}$$

for some $\beta(0 \leq \beta < 1)$. We denote by $S_w^*(\beta)$ the class of all starlike functions of order β. Similarly , a function f in S_w is said to be convex of order β if and only if

$$\text{Re}\left(-1 - \frac{(z-w)\, f''(z)}{f'(z)}\right) > \beta, \qquad ((z-w) \in U) \tag{6}$$

for some $\beta(0 \leq \beta < 1)$. We denote by $CV_w(\beta)$ the class of all convex functions of order β. We note that the class $S_0^*(\beta)$ and various other subclasses of $S_w^*(\beta)$ have been studied rather extensively by Nehari and Netanyahu [25], Acu and Owa [15], Clunie [12], Pommerenke[[7],[8]], Miller[13], Royster [24], and others (cf., e.g.,Bajpai[22], Goel and Sohi[21], Mogra et al [17], Uralegaddi and Ganigi [4], Cho et al[19], Aouf [16], and Uralegaddi and Somanatha ([5],[6]); see also Duren ([20] , pp.29 and 137), Srivastava and Owa [[11] , pp.86 and 429] and Ghanim and Darus [[9],[10]].

For the function f in the class S_w, we define

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = (z-w)f'(z) + \frac{2}{z-w},$$

$$I^2 f(z) = (z-w)\left(I^1 f(z)\right)' + \frac{2}{z-w},$$

and for k=1, 2, 3,... we can write

$$I^k f(z) = (z-w)\left(I^{k-1} f(z)\right)' + \frac{2}{z-w}$$

$$= \frac{1}{z-w} + \sum_{n=1}^{\infty} n^k a_n (z-w)^n. \tag{7}$$

The differential operator I^k studied extensively by Ghanim and Darus [[9],[10]] and in the case $w = 0$ was given by Frasin and Darus [3].

With the help of the differential operator I^k, we define the class $S_w^*(k, \beta)$ as follows:

Definition 1.1 *The function $f(z) \in S_w$ is said to be a member of the class $S_w^*(k, \beta)$ if it satisfies*

$$\left| \frac{(z-w)\left(I^k f(z)\right)'}{I^k f(z)} + 1 \right| < \left| \frac{(z-w)\left(I^k f(z)\right)'}{I^k f(z)} + 2\beta - 1 \right|$$

$(k \in N_0 = N \cup 0)$, *for some* $\beta (0 \leq \beta < 1)$ *and for all* $z (0 \leq z < 1)$ *in* D.

It is easy to check that $S_w^*(0, \beta)$ is the class of starlike functions of order β and $S_w^*(0, 0)$ gives the starlike functions for all $z \in D$.

Let us write

$$S_w^*[k, \beta] = S_w^*(k, \beta) \cap S_w \tag{8}$$

where S_w is the class of functions of the form (1.4) that are analytic and univalent in D.

Let H be a Hilbert space on the complex field. Let A be an operator on H. For a complex analytic function f on the unit disk D, we denoted $f(A)$, the operator on H defined by Riesz-Dunford integral [18]

$$f(A) = \frac{1}{2\pi i}\int_C f(z)\,(zI - A)^{-1}dz$$

where I is the identity operator on H. C is a positively oriented simple closed rectifiable contour lying in D and containing the spectrum of A in its interior domain [14]. The conjugate operator of A is A^*.

A function f given by (1.4) is in the class $S_w^*[k,\beta,H]$ if it is satisfies the condition:

$$\left\|A\left(I^k f(A)\right)' + I^k f(A)\right\| < \left\|A\left(I^k f(A)\right)' + (2\beta-1)\,I^k f(A)\right\|,$$

for $0 \leq \beta < 1$ and for all operator A with $\|A\| < 1$ and $A \neq \Theta$ (Θ is the zero operator on H).

In the present paper, we obtain coefficient estimates and distortion theorem for $S_w^*[k,\beta,H]$.

2 Main Results

Our first result provides a sufficient condition for a function f analytic in D to be in $S_w^*[k,\beta,H]$.

Theorem 2.1 *A function f given by (1.4) is in the class $S_w^*[k,\beta,H]$ for all proper contraction A with $A \neq \Theta$ if and only if*

$$\sum_{n=1}^{\infty} n^k\,(n+\beta)\,a_n \leq (1-\beta) \tag{9}$$

for $0 \leq \beta < 1$.

Proof: Assume that (2.1) holds, we have

$$\left\|A\left(I^k f(A)\right)' + I^k f(A)\right\| - \left\|A\left(I^k f(A)\right)' + (2\beta-1)\,I^k f(A)\right\|$$

$$= \left\|\sum_{n=1}^{\infty} n^k\,(n+1)\,a_n A^n\right\| - \left\|\frac{2(\beta-1)}{A} + \sum_{n=1}^{\infty} n^k\,(n+2\beta-1)\,a_n A^n\right\|$$

$$\leq \left\| \sum_{n=1}^{\infty} n^k \left(n+1\right) a_n A^{n+1} \right\| - \left\| 2(1-\beta) - \sum_{n=1}^{\infty} n^k \left(n+2\beta-1\right) a_n A^{n+1} \right\|$$

$$\leq \sum_{n=1}^{\infty} 2n^k \left(n+\beta\right) a_n - 2(1-\beta) \leq 0$$

Hence f is in the class $S_w^*[k, \beta, H]$.

Conversely, suppose that

$$\left\| A \left(I^k f(A)\right)' + I^k f(A) \right\| < \left\| A \left(I^k f(A)\right)' + (2\beta-1)\, I^k f(A) \right\|,$$

so that

$$\left\| \sum_{n=1}^{\infty} n^k \left(n+1\right) a_n A^{n+1} \right\| < \left\| 2(\beta-1) + \sum_{n=1}^{\infty} n^k \left(n+2\beta-1\right) a_n A^{n+1} \right\|$$

Selecting $A = eI (0 < e < 1)$ in above inequality, we have

$$\frac{\sum\limits_{n=1}^{\infty} n^k \left(n+1\right) a_n e^{n+1}}{2(1-\beta) - \sum\limits_{n=1}^{\infty} n^k \left(n+2\beta-1\right) a_n e^{n+1}} < 1 \tag{10}$$

Upon clearing denominator in (2.2) and letting $e \rightarrow 1 (0 < e < 1)$, we get

$$\sum_{n=1}^{\infty} n^k \left(n+1\right) a_n < 2(1-\beta) - \sum_{n=1}^{\infty} n^k \left(n+2\beta-1\right) a_n$$

which implies that

$$\sum_{n=1}^{\infty} n^k \left(n+\beta\right) a_n \leq (1-\beta)$$

for $0 \leq \beta < 1$. This completes the proof of the theorem.

Corollary 2.2 *If f given by (1.4) is in the class $S_w^*[k, \beta, H]$ for $0 \leq \beta < 1$ and $k \in N_0$ then*

$$a_n \leq \frac{(1-\beta)}{n^k \left(n+\beta\right)}, \qquad (n \geq 1). \tag{11}$$

Corollary 2.3 *If f given by (1.4) is in the class $S_w^*[k, \beta, H]$ for $0 \leq \beta < 1$ and $k \in N_0$ then*

$$na_n \leq \frac{(1-\beta)}{n^{k-1} \left(n+\beta\right)}, \qquad (n \geq 1). \tag{12}$$

Next, we consider the growth and distortion properties as the following:

Theorem 2.4 *If the function* f *given by (1.4) in the class* $S_w^*[k,\beta,H]$ *for* $0 \leq \beta < 1, \|A\| < 1$ *and* $\|A\| \neq \Theta$ *then*

$$\frac{1}{\|A\|} - \frac{(1-\beta)}{(1+\beta)}\|A\| \leq \|f(A)\| \leq \frac{1}{\|A\|} + \frac{(1-\beta)}{(1+\beta)}\|A\|. \tag{13}$$

The result is sharp for the function

$$f_1(z) = \frac{1}{z-w} + \frac{(1-\beta)}{(1+\beta)}(z-w).$$

Proof: In view of Theorem 2.1, we have

$$(1+\beta)\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} n^k (n+\beta) a_n \leq (1-\beta)$$

which gives

$$\sum_{n=1}^{\infty} a_n \leq \frac{(1-\beta)}{(1+\beta)}$$

Hence we have

$$\|f(A)\| \geq \frac{1}{\|A\|} - \|A\|\sum_{n=1}^{\infty} a_n \geq \frac{1}{\|A\|} - \frac{(1-\beta)}{(1+\beta)}\|A\|$$

also we have

$$\|f(A)\| \leq \frac{1}{\|A\|} + \|A\|\sum_{n=1}^{\infty} a_n \leq \frac{1}{\|A\|} + \frac{(1-\beta)}{(1+\beta)}\|A\|.$$

Hence the theorem.

Theorem 2.5 *If the function* f *given in the (1.4) in the class* $S_w^*[k,\beta,H]$, *for* $0 \leq \beta < 1$, $\|A\| < 1$ *and* $\|A\| \neq \Theta$ *then*

$$\frac{1}{\|A\|^2} - \frac{(1-\beta)}{(1+\beta)} \leq \|f'(A)\| \leq \frac{1}{\|A\|^2} + \frac{(1-\beta)}{(1+\beta)}. \tag{14}$$

The result is sharp for the function

$$f_1(z) = \frac{1}{z-w} + \frac{(1-\beta)}{(1+\beta)}(z-w)^n.$$

Proof: We have

$$f'(z) = \frac{1}{(z-w)^2} + \sum_{n=1}^{\infty} n a_n (z-w)^{n-1}$$

Since, $f \in S_w^*[k, \beta, H]$ we have

$$\|f'(A)\| \leq \frac{1}{\|A\|^2} + \sum_{n=1}^{\infty} n a_n \|A\|^{n-1}$$

In view of Theorem 2.1, we have

$$(1+\beta) \sum_{n=1}^{\infty} n a_n \leq (1-\beta)$$

which gives

$$\sum_{n=1}^{\infty} n a_n \leq \frac{(1-\beta)}{(1+\beta)}.$$

Consequently, we have

$$\|f'(A)\| \leq \frac{1}{\|A\|^2} + \sum_{n=1}^{\infty} n a_n \leq \frac{1}{\|A\|^2} + \frac{(1-\beta)}{(1+\beta)}$$

and

$$\|f'(A)\| \geq \frac{1}{\|A\|^2} - \sum_{n=1}^{\infty} n a_n \|A\|^{n-1}.$$

Thus, we have

$$\|f'(A)\| \geq \frac{1}{\|A\|^2} - \sum_{n=1}^{\infty} n a_n \geq \frac{1}{\|A\|^2} - \frac{(1-\beta)}{(1+\beta)}.$$

This completes the proof of the theorem.

Theorem 2.6 *Let*

$$f_0(z) = \frac{1}{z-w}$$

and

$$f_n(z) = \frac{1}{z-w} - \frac{(1-\beta)}{n^k (n+\beta)} (z-w)^n, \qquad n = 1, 2, 3, ..., (z-w) \in D.$$

Then $f \in S_w^*[k, \beta, H]$ *if and only if it can be expressed in the form*

$$f(z) = \lambda_0 f_0(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z)$$

where $\lambda_n \geq 0$ *and* $\lambda_0 + \sum_{n=1}^{\infty} \lambda_n = 1$.

Proof: Let us assume that

$$f(z) = \lambda_0 f_0(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z)$$

$$= \frac{1}{z-w} + \sum_{n=1}^{\infty} \frac{(1-\beta)\lambda_n}{n^k (n+\beta)} (z-w)^n .$$

Then we have

$$\sum_{n=1}^{\infty} \frac{n^k (n+\beta)}{(1-\beta)} \lambda_n . \frac{(1-\beta)}{n^k (n+\beta)} = \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1.$$

Hence $f \in S_w^*[k, \beta, H]$.

Conversely, we assume that f given by (1.4) is in the class $S_w^*[k, \beta, H]$. From Corollary 2.2, we have

$$a_n \leq \frac{(1-\beta)}{n^k (n+\beta)}.$$

we may set

$$\lambda_n = \frac{n^k (n+\beta)}{\alpha(1-\beta)} a_n, \qquad (n \geq 1; k \in N_0)$$

and

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n$$

we have

$$f(z) = \lambda_0 f_0(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z)$$

This completes the proof of the theorem.

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References

[1] A. W. Goodman, On uniformly starlike functions, *J. Math. Anal. Appl.*, **155** (1991), 364-370.

[2] A. W.Goodman, On uniformly convex functions, *Ann. Polon. Math.*, **56** (1991), no.1, 87-92.

[3] B. A. Frasin and M.Darus, On certain meromophic functions with positive coefficients, *South East Asian Bull. Math.* **28** (2004), 615-623.

[4] B. A. Uralgaddi and M.D. Ganigi, A certain class of meromorphic univalent functions with positive coefficients, *Pure Appl. Math. Sci.*, **26** (1987), 75-81.

[5] B. A. Uralgaddi and C. Somanatha, New criteria for meromorphic starlike univalent functions, *Bull. Austral. Math. Soc.*, **43**(1991), 137-140.

[6] B. A. Uralgaddi and C. Somanatha, Certain differential operators for meromorphic functions, *Houston J. Math.* **17** (1991), 279-284.

[7] Ch. Pommerenke, On meromorphic starlike functions, *Pacific J. Math.*, **13** (1963), 221-235.

[8] Ch. Pommerenke, Über einige klassen meromorfer schlichter funktionen, *Math. Zeitschr.*, **78** (1962), 263-284.

[9] F. Ghanim and M. Darus, On certain class of analytic function with fixed second positive coefficient, *Int. Jour. Math. Anal.*, **2** No.2, (2008), 55-66.

[10] F. Ghanim, M. Darus and S. Sivasubramanian, On new subclass of analytic univalent function, *Inter. J. Pure Appl. Math.,* **40**, No.3, (2007), 307-319.

[11] H. M. Srivastava and S. Owa, (Eds.), "Current Topics In Analytic Functions Theory", *World Scientific*, Singapore/ New Jersey/ London/ Hong Kong, 1992.

[12] J. Clunie, On Meromorphic Schlicht Functions, *J. London Math. Soc.*, **34** (1959), 215-216.

[13] J. Miller, Convex meromorphic mappings and related functions, P̂roc. Amer. Math. Soc. **25** (1970), 220-228.

[14] K. Fan, Analytic functions of a proper contractions, *Math. Z.*, **160** (1978), 275-290.

[15] M. Acu and S. Owa, On some subclass of univalent functions, *J. Inequal. Pure Appl. Math.*, **6** (2005), 1-6.

[16] M. K. Aouf, On a certain class of meromorphic univalent functions with positive coefficient, *Rend. Mat. Appl.*, **7** 11 (1991), 209-219.

[17] M. L. Mogra, T.R. Reddy, and O. P. Juneja, Meromorphic univalent functions with positive coefficients, *Bull. Austral. Math. Soc.* **32**(1985), 161-176.

[18] N. Dunford and J. T. Schwartz, Linear operators, Part I, General Theory, New York-London, Intersciences, 1985.

[19] N. E. Cho, S. H. Lee, and S.Owa, A class of meromorphic univalent functions with positive coefficients, *Kobe J. Math.*, **4** (1987), 43-50.

[20] P. L. Duren, "Univalent Functions" Grundlehren Der Mathematischen Wissenschafen, Vol. 259, *Springer- Verlag*, Newyork/Berlin/Heidelberg/Tokyo, 1983.

[21] R. M. Goel and N. S. Sohi, On a class of meromorphic functions, *Glas. Mat. Ser. III* **17**(37), (1982), 19-28.

[22] S. K. Bajpai, A note on a class of meromorphic univalent functions, *Rev. Roumaine Math. Pure Appl.*, **22** (1977), 295 -297.

[23] S. Kanas and F. Ronning, Uniformly starlike and convex function and other related classes of univalent functions, *Ann. Univ.Mariae Curie-Sklodowaska* section A, **53** (1991), 95-105.

[24] W. C. Royster, Meromorphic starlike multivalent functions, *Trans. Amer. Math. Soc.*, **107** (1963), 300-308.

[25] Z. Nehari and E. Netanyahu, On the coefficients of meromorphic schlicht functions, *Proc. Amer. Math. Soc.*, **8** (1957), 15-23.

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