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On Uniform Statistical Convergence of Random Variable Sequences

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Abstract. In this study, we introduce a notion of uniform statistical convergence of random variable sequences. We show that a continuous function conserves the limit of statistical convergence of real-number sequences. Furthermore we introduce the concept of a uniform statistically Cauchy sequence for random variables and prove that it is equivalent to uniform statistical convergence of sequences of random variables.

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1. Introduction

Let Ω be an '*abstract space'*, namely a nonempty set of elements to be called '*points*' and denoted generally by ω and let Γ be a Borel field on Ω. A real-valued random variable is a function X, whose domain is a set Δ in Γ and whose range is contained in **R** = $(-\infty, +\infty)$ such that for each B in $\Im^*$ (the extended Borel field) [1].

As we know, a sequence of numbers ω_k tends to a statistical limit ω, if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \{ \text{the number of } k \leq n : | \omega_k - \omega | \geq \varepsilon \} = 0,$$

i.e.,

$$| \omega_k - \omega | < \varepsilon \ \text{ a.a.k.}$$

In this case, we write st-lim$\omega_k = \omega$ [2].

Consider now the sequence $X_1(\omega)$, $X_2(\omega)$, ..., $X_n(\omega)$, ... of random variables. We have a sequence of numbers

$$X_1(\omega), X_2(\omega), ..., X_k(\omega), ... \tag{1}$$

for each experimental outcome ω.

If it happens that (1) converges for all ω, then we say that (1) converges uniformly on Δ. The limit of each sequence will be depend on ω; that is, if $X_k(\omega) \to X(\omega)$ for $k \to \infty$, then $X(\omega)$ is, in general, a random variable.

The above convergence mode is too restrictive. In many cases, (1) may not converge uniformly. However, convergence in such a strong sense is almost completely uncharacteristic of probabilistic models. Now, let us give an example.

Example 1: In a table, four signal producers A, B, C and D are examined. We will consider the appearance of signals {A, D} where $\Omega = \{A, B, C, D\}$ and Borel field is $\Gamma = \{ \phi, \Omega, \{A, D\}, \{B, C\}\}$. Sequence of random variable defined with respect to the result of the experiment is

$$X_k: \{A, D\} \to \mathbf{R}$$

$$\omega \to X_k(\omega) = \begin{cases} 1, & \text{if } \omega \text{ occurs at the } k = i^2 \text{ th trial} (i = 1, 2, ...) \\ 0, & \text{otherwise} \end{cases}$$

and it can be formed as (1, 0, 0, 1, 0, 0, 0, 0, 1, 0, ...).

This sequence is not uniformly convergent. Thus we may introduce a different kind of convergence.

2. Uniform statistical convergence of random variable sequences

In this section, we first introduce uniform statistical convergence and then uniform statistically Cauchy sequence for random variables.

Definition 1: Let $(X_k(\omega))$ be a sequence of random variable defined on Δ. The sequence $(X_k(\omega))$ is uniform statistically convergent to $X(\omega)$, if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \{\text{the number of } k \le n : |X_k(\omega) - X(\omega)| \ge \varepsilon \text{ for all } \omega \in \Delta\} = 0,$$

i.e., for all $\omega \in \Delta$,

$$|X_k(\omega) - X(\omega)| < \varepsilon \text{ a.a.k.}$$

In this case, we write st-lim $X_k(\omega) = X(\omega)$ uniformly on Δ.

Let us consider Example 1 again. We have

$$\lim_{n\to\infty} \frac{1}{n} \{\text{the number of } k \le n : |X_k(\omega) - X(\omega)| \ge \varepsilon \text{ for all } \omega \in \{A, D\}\} \le \lim_{n\to\infty} \frac{\sqrt{n}}{n} = 0$$

where $X(\omega) = 0$ so the sequence $(X_k(\omega))$ is not uniformly convergent but uniform statistically convergent.

Theorem 1: If $\lim_{k\to\infty} X_k(\omega) = X(\omega)$ uniformly on Δ, then st-lim$X_k(\omega) = X(\omega)$ uniformly on Δ.

Proof: The proof is trivial.

Theorem 2: Let $(X_k(\omega))$ and $(Y_k(\omega))$ be two sequences of random variables on Δ. If st-lim $X_k(\omega) = X(\omega)$ and st-lim$Y_k(\omega) = Y(\omega)$ uniformly on Δ, then

st-lim$(\alpha X_k(\omega) + \beta Y_k(\omega)) = \alpha X(\omega) + \beta Y(\omega)$ uniformly on Δ,

where $\alpha, \beta \in \mathbf{R}$.

Proof: The proof is similar to that of Theorem 2.1 in [3].

The following theorem is our main result which characterizes uniform statistical convergence of any function sequence.

Theorem 3: Let X_k and X be continuous random variables on $I = [a, b] \subset \mathbf{R}$ for each $k \in \mathbf{N}$. Then st-lim $X_k(\omega) = X(\omega)$ uniformly on I if and only if st-lim $c_k = 0$ where

$$c_k = \max_{\omega \in I} |X_k(\omega) - X(\omega)|.$$

Proof: The proof is similar to that of Theorem 2.2 in [4].

The following theorem shows that a continuous function conserves the limit of statistical convergence of a real number sequence.

Theorem 4: If st-lim $\omega_k = \ell$ and $X_m(\omega)$, defined for all real ω, is continuous at $\omega = \ell$ for each fixed $m \in \mathbf{N}$, then st-lim $X_m(\omega_k) = X_m(\ell)$ uniformly on $\mathbf{R}$ for each fixed $m \in \mathbf{N}$.

Proof: Indeed, given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\omega - \ell| < \delta$ implies $|X_m(\omega) - X_m(\ell)| < \varepsilon$ for each fixed $m \in \mathbf{N}$. But then $|X_m(\omega) - X_m(\ell)| \ge \varepsilon$

implies $|\omega - \ell| \geq \delta$ and in particular, $|X_m(\omega_k) - X_m(\ell)| \geq \varepsilon$ implies $|\omega_k - \ell| \geq \delta$. Thus, for fixed $m \in \mathbf{N}$,

$$\{k \leq n : |X_m(\omega_k) - X_m(\ell)| \geq \varepsilon\} \subset \{k \leq n : |\omega_k - \ell| \geq \delta\}$$

and therefore

$$\lim_{n\to\infty} \frac{1}{n}\{\text{the number of } k \leq n : |X_m(\omega_k) - X_m(\ell)| \geq \varepsilon\} \leq \lim_{n\to\infty} \frac{1}{n}\{\text{the number of } k \leq n : |\omega_k - \ell| \geq \delta\} = 0.$$

This proves that st-lim $X_m(\omega_k) = X_m(\ell)$ uniformly on $\mathbf{R}$, for each fixed $m \in \mathbf{N}$.

Definition 2: Let $\{X_k\}$ be a sequence of random variables on a set Δ. The sequence $\{X_k\}$ is a uniform statistically Cauchy sequence provided that for each $\varepsilon > 0$ there exists a number N (= N (ε))
such that

$$|X_k(\omega) - X_N(\omega)| < \varepsilon \quad \text{a.a. } k \quad \text{for all } \omega \in \Delta,$$

i.e.

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |X_k(\omega) - X_N(\omega)| \geq \varepsilon \text{ for all } \omega \in \Delta\}| = 0.$$

Gökhan and Güngör [3] obtained an equivalent criterion for pointwise statistical convergence of a sequence of real-valued functions, similar to the Cauchy criterion for convergence. We give a similar criterion for uniform statistically convergent for random variables as follows:

Theorem 5: Let $\{X_k\}$ be a sequence of random variables defined on a set Δ. The following statements are equivalent:

i) $\{X_k\}$ is a uniform statistically convergent sequence on Δ ;

ii) $\{X_k\}$ is a uniform statistically Cauchy sequence on Δ ;

iii) $\{X_k\}$ is a sequence of random variables for which there is a uniformly convergent sequence $\{Y_k\}$ such that $X_k(\omega) = Y_k(\omega)$ a.a. k for all $\omega \in \Delta$.

Proof: The proof is similar to that of Theorem 2.5 in [4].

Corrollary: If $\{X_k\}$ is a sequence of functions such that st-lim $X_k(\omega) = X(\omega)$ uniformly on Δ, then $\{X_k\}$ has a subsequence $(X_{k(i)}(\omega))$ such that $\lim_{i\to\infty} X_{k(i)}(\omega) = X(\omega)$ uniformly on Δ.

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