Some Oscillation Theorems for Systems of Partial Differential Equations with Deviating Arguments and a Note on Impulsive Hyperbolic Equation

Chen Ning

Department of mathematics and physics
Southwest university of Science and Technology
Mianyang 621010, Sichuan, P.R. China
cy783@yahoo.com.cn

Abstract

In this paper, we give some results of the oscillations criteria of the solution for some higher-order equations with deviating arguments, and note of the impulsive hyperbolic equations. We get some new conclusions, which generalize the results in [4] and [5].

Mathematics Subject classification: 35R10, 35B05

Keywords: Oscillation; Delay Hyperbolic equation; Impulses

1 Introduction and Lemma

Recently, the oscillation of solutions for higher-order partial differential equations with deviating arguments is widely usually discussed (see [2]-[4]etc). Aside from their intrinsic interest, oscillation of solutions is very important in the domain of physics (this things are interesting with some example). In this paper, we consider a more generalized higher-order equation.

Now in the direction of [2], our conclusions extend and complete the previous results in [1]-[5].

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \) having sufficiently smooth boundary \( \partial \Omega \), and \( n \) be an even positive integer number, \( (x,t) \in \Omega \times [0, \infty) \subseteq G \), Let
We consider the oscillation of solutions of systems:

\[ \frac{\partial^{n} u_j(x,t)}{\partial t^{n}} + \frac{\partial^{n-1} u_j(x,t)}{\partial t^{n-1}} = P_j(x,t) + P_j(x,t - \rho_k(t)) \quad \text{(*)} \]

\[ - \sum_{j=1}^{m} p_{ij}(x,t) u_j(x,t - \sigma(t)), i = 1, 2, \ldots, m \]

(\text{where} \quad P_j(x, t- \rho_k(t)) = \sum_{k=1}^{s} [a_{ik}(t) \Delta + b_{ik}(t) \Delta^2 + c_{ik}(t) \Delta^3] u_i(x, t - \rho_k(t)), \gamma\]

denotes the derivative in outward normal direction on \( \partial \Omega \), and \( u_i(x,t) \) is defined to be real function, and \( \Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}, \Delta^2 = \Delta(\Delta), \Delta^3 = \Delta(\Delta^2), \ldots. \)

Let \( g_i(x,t) \) be a non-negative continuous function in \( \partial \Omega \times [0, \infty) \), and satisfying two conditions:

\[ \frac{\partial u_i(x,t)}{\partial \gamma} + g_i(x,t) u_i(x,t) = 0, \quad \frac{\partial \Delta u_i(x,t)}{\partial \gamma} + g_i(x,t) u_i(x,t) = 0, \]

\[ \frac{\partial \Delta^2 u_i(x,t)}{\partial \gamma} + g_i(x,t) u_i(x,t) = 0, \quad (x,t) \in \partial \Omega \times [0, \infty), (i = 1, 2, \ldots, m) \quad (Q_1) \]

and

\[ u_i(x,t) = 0, \Delta u_i(x,t) = 0, \Delta^2 u_i(x,t) = 0, \quad (x, t) \in \partial \Omega \times [0, \infty) (i = 1, 2, \ldots, m). \quad (Q_2) \]

By the definition and some prescribe in [2], we assume that it satisfies (H):

(\text{G}_1) \quad \sigma, \rho_k \in C([0, \infty), [0, \infty)), \text{and} \quad \lim_{t \to \infty} (t - \sigma(t)) = \infty,

\[ \lim_{t \to \infty} (t - \rho_k(t)) = \infty, k = 1, 2, \ldots, s. \]

(\text{G}_2) \quad p_{ij}(x, t) \in C(\overline{\Omega}, R), p_{ij}(x, t) > 0, p_{ij}(t) = \min_{x \in \overline{\Omega}} p_{ij}(x,t), p_{ij}(t) = \sup_{x \in \overline{\Omega}} |p_{ij}(x,t)|,

\[ Q(t) = \min_{1 \leq j \leq m} \left\{ p_{ij}(t) - \sum_{j=1, j \neq i}^{m} p_{ij}(t) \right\} \geq 0, \quad i = 1, 2, \cdots, m; j = 1, 2, \cdots, m. \]
Oscillation theorems for systems of PDE

We will give two theorems to extend some results which are similar to in [2].

Remark If we take \( b_i(t), c_i(t) \equiv 0 \) in (*), then we get theorem 1 in [2], and

From the last part of proof of theorem 1 in [2], we have the following two lemmas (lemma 1 and lemma 2).

Now we list following lemma (by \( V_i(t) = \int_{\Omega} Z_i(x,t)dx \), and \( V(t) = \sum_{i=1}^{m} V_i(t) \))

**Lemma 1** If

\[
\frac{d^n}{dt^n} \left( \int_{\Omega} Z_i(x,t)dx \right) + \frac{d^{n-1}}{dt^{n-1}} \left( \int_{\Omega} Z_i(x,t)dx \right) \leq -p_{ii}(t) \int_{\Omega} Z_i(x,t - \sigma(t))dx + \sum_{j=1, j \neq i}^{m} p_{ij}(t) \int_{\Omega} Z_j(x,t - \sigma(t))dx, t \geq t_i, i = 1, 2, \ldots, m. \tag{1}
\]

then

\[
V^{(n)}(t) + V^{(n-1)}(t) + Q(t) V(t - \sigma(t)) \leq 0, \quad t \geq t_i.
\]

Proof From \( V_i(t) = \int_{\Omega} Z_i(x,t)dx \), and \( V(t) = \sum_{i=1}^{m} V_i(t) \), and the last part of proof of theorem 1 in [2], it is easy to get it. So the proof of the lemma is omitted.

**Lemma 2** If

\[
V^{(n)}(t) + V^{(n-1)}(t) + Q(t) V(t - \sigma(t)) \leq 0, \quad t \geq t_i
\]

then \( \int_{t_i}^{\infty} Q(t)dt < \infty \)

Proof It is as same as the last part of proof of theorem 2 in [2].

In the following part, we will give out oscillation criteria of theorems for system (*)-(Q_1).

2 Several theorems

**Theorem 1** If \( \int_{t_0}^{\infty} Q(t)dt = \infty \), \( t_0 > 0 \), then all solutions of the system (*)-(Q_1) are oscillation in G.

Proof We suppose to the contrary there exists a non-oscillation solution
\[ u(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_m(x,t)) \] of the system \((\ast)-(Q_1)\) for some \(0 \leq t_0 \leq t, |u_i(x,t)| > 0\). Let \(\delta_i = \text{sign} u_i(x,t), i = 1,2,\ldots, m\) and \(Z_i(x,t) = \delta_i u_i(x,t)\).

Then we have \(Z_i(x,t) > 0\), where \((x,t) \in \Omega \times [t_0, \infty)\).

From condition \((G_1)\), we easily know that there exists \(t_1 \geq t_0\) such that when \(t \geq t_1\), we have \(Z_i(x,t) > 0, Z_i(x, t- \rho_k(t)) > 0, Z_i(x, t- \sigma(t)) > 0\). where \((x, t) \in \Omega \times [t_1, \infty), i = 1,2,\ldots, m; k = 1,2,\ldots, s\).

Integrating both side of \((\ast)\) for \(x\) over \(\Omega\), we have that

\[
\frac{d^n}{dt^n} \left( \int_{\Omega} u(x,t)dx \right) + \frac{d^{n-1}}{dt^{n-1}} \left( \int_{\Omega} u(x,t)dx \right) = \int_{\Omega} P_3(x,t)dx + \int_{\Omega} P(x,t- \rho_k(t))dx
\]

\[-\sum_{j=1}^{m} P_{ij}(x,t) \int_{\Omega} u_j(x,t- \sigma(t))dx, t \geq t_1, i = 1,2,\ldots, m.\] That is

\[
\frac{d^n}{dt^n} \left( \int_{\Omega} Z_i(x,t)dx \right) + \frac{d^{n-1}}{dt^{n-1}} \left( \int_{\Omega} Z_i(x,t)dx \right) = \int_{\Omega} P_3(x,t)dx + \int_{\Omega} P(x,t- \rho_k(t))dx
\]

\[-\sum_{j=1}^{m} P_{ij}(x,t) \int_{\Omega} u(x,t- \sigma(t))dx, t \geq t_1, i = 1,2,\ldots, m.\] (3)

Similar to the proof of theorem 1, by Green identity and boundary value conditions \((Q_1)\), we have that

\[
\int_{\Omega} \Delta^2 Z_i(x,t)dx = -\int_{\partial\Omega} \frac{\partial \Delta Z_i(x,t)}{\partial n}ds - \int_{\Omega} g_i(x,t) Z_i(x,t)ds \leq 0, \text{ and}
\]

\[
\int_{\Omega} \Delta^2 Z_i(x,t- \rho_k(t))dx = -\int_{\partial\Omega} \frac{\partial \Delta Z_i(x,t- \rho_k(t))}{\partial n}ds
\]

\[= -\int_{\partial\Omega} g_i(x,t- \rho_k(t))Z_i(x,t- \rho_k(t))ds \leq 0.\]

\[
\int_{\Omega} \Delta^2 Z_i(x,t)dx = -\int_{\partial\Omega} \frac{\partial Z_i(x,t)}{\partial n}ds - \int_{\Omega} g_i(x,t)Z_i(x,t)ds \leq 0, \text{ and also that}
\]
Oscillation theorems for systems of PDE

\[
\int_{\Omega} \Delta^2 Z_i(x, t - \rho_k(t))dx = \int_{\Omega} \frac{\partial \Delta Z_i(x, t - \rho_k(t))}{\partial n} ds
\]

\[
= -\int_{\Omega} g_i(x, t - \rho_k(t))Z_i(x, t - \rho_k(t))ds \leq 0.
\]

and \( \int_{\Omega} \Delta Z_i(x, t)dx = \int_{\Omega} \frac{\partial Z_i(x, t)}{\partial n} ds = -\int_{\Omega} g_i(x, t)Z_i(x, t)ds \leq 0, \ i = 1, 2, \cdots, m. \)

\[
\int_{\Omega} \Delta Z_i(x, t - \rho_k(t))dx = \int_{\Omega} \frac{\partial Z_i(x, t - \rho_k(t))}{\partial n} ds
\]

\[
= -\int_{\Omega} g_i(x, t - \rho_k(t))Z(x, t - \rho_k(t))ds \leq 0.
\]

Thus from above stating and combing conditions (G_2), (3) holds. Now by lemma 1 and lemma 2, we have \( \int_t^\infty Q(t)dt < \infty \), which is contradictory to the condition of theorem . Then this theorem is proved.

**Corollary** If the differential inequality (2) has no eventually positive solution, then all solution of (*)-(Q_1) are oscillation in \( G \) (the same as corollary 2 in [2]).

It is well known that the first eigenvalue \( \lambda_0 \) of the problem

\[
\Delta \varphi + \lambda \varphi = 0 \quad \text{in} \quad \Omega, \quad \varphi = 0 \quad \text{on} \quad \partial \Omega
\]

is positive and the corresponding eigenfunction \( \varphi \) is positive in \( \Omega \).

**Lemma 3** (see the proof of theorem 2 in [2]) Assume that

\[
\frac{d^n}{dt^n} \left( \int_{\Omega} Z_i(x, t)\varphi(x)dx \right) + \frac{d^{(n-1)}}{dt^{(n-1)}} \left( \int_{\Omega} Z_i(x, t)\varphi(x)dx \right)
\]

\[
\leq -p_i(t)\int_{\Omega} Z_i(x, t - \sigma(t))\varphi(x)dx + \sum_{j=1, j\neq i}^m P_j(t)\int_{\Omega} Z_j(x, t - \sigma(t))\varphi(x)dx, \ t \geq t_1, (3)'.
\]

Then \( V_{i}^{(n)}(t) + V_{i}^{(n-1)}(t) + Q(t)V(x, t - \sigma(t)) \leq 0, \ t \geq t_1. \)
**Theorem 2**  If \( \int_{t_0}^{\infty} Q(t) dt = \infty, t_0 > 0 \), then all solutions of the systems (*)-(Q_2) are oscillation in \( G \).

Proof. Suppose to the contrary. Then there exists a non-oscillation solution: \( u(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_m(x,t)) \) of system (*)-(Q_2) in the domain \( \Omega \times [t_0, +\infty) \) for some \( t_0 > 0 \). For convenience and simplicity, we may take as

\[
0 \leq t_0 \leq t, \quad |u_i(x,t)| > 0, \quad (i = 1,2,\ldots, m) \quad \text{and} \quad Z_i(x,t) = \delta_i u_i(x,t),
\]

and \( \delta_i = \text{sign} u_i(x,t) \). Then we have \( Z_i(x,t) > 0 \). From (G_1) there exists \( t_1 \geq t_0 \), such that when \( t \geq t_1 \), we have \( Z_i(x,t) > 0, \quad Z_i(x,t-\rho_i(t)) > 0, \quad i = 1,2,\ldots, m \).

\[
k = 1,2,\ldots, s . \quad (x,t) \in \Omega \times [t_1, \infty).
\]

Multiplying both sides of (*) by \( \varphi(x) \), and integrating for \( x \) on \( \Omega \), we get

\[
\frac{d^n}{dt^n} \left( \int_{\Omega} u_i(x,t) \varphi(x) dx \right) + \frac{d^{n-1}}{dt^{n-1}} \left( \int_{\Omega} u_i(x,t) \varphi(x) dx \right) = \int_{\Omega} P_3(x,t) \varphi(x) dx + \int_{\Omega} P_1(x,t) \varphi(x) dx
\]

Therefore, we have that

\[
\frac{d^n}{dt^n} \left( \int_{\Omega} Z_i(x,t) \varphi(x) dx \right) + \frac{d^{n-1}}{dt^{n-1}} \left( \int_{\Omega} Z_i(x,t) \varphi(x) dx \right)
\]

\[
= a_i(t) \int_{\Omega} \Delta Z_i(x,t) \varphi(x) dx + \cdots + c_i(t) \int_{\Omega} \Delta^3 Z_i(x,t) \varphi(x) dx
\]

\[
+ \sum_{k=1}^m a_{ik}(t) \int_{\Omega} Z_j(x,t-\rho_k(t)) \varphi(x) dx + \cdots + \sum_{k=1}^m c_{ik}(t) \int_{\Omega} \Delta^3 Z(x,t-\rho_k(t)) \varphi(x) dx
\]

\[
- \sum_{j=1}^m \frac{\delta_j}{\delta_j} \int_{\Omega} P_j(x,t) Z_i(x,t-\sigma(t)) \varphi(x) dx, \quad t \geq t_1.
\]

From Green identity and boundary value conditions (Q_2) we obtain that

\[
\int_{\Omega} \Delta Z_i(x,t) \varphi(x) dx = \lambda_0 \int_{\Omega} Z_i(x,t) \varphi(x) dx \leq 0, \cdots, \leq 0,
\]
Oscillation theorems for systems of PDE

\[ \int_{\Omega} \Delta Z_j (x, t - \rho_k (t)) \varphi (x) dx \leq 0, \cdots, \]
and
\[ \int_{\Omega} \Delta^2 Z_j (x, t - \rho_k (t)) \varphi (x) dx \leq 0, \cdots, \int_{\Omega} \Delta^3 Z_j (x, t - \rho_k (t)) \varphi (x) dx \leq 0, \]
\[ t \geq t_1, i = 1, 2, \cdots, m. \]

Then (3)' holds. By lemma 3 and lemma 2, we have \( \int_0^\infty Q(t) dt < \infty \), which is contradictory to the condition of the theorem. Then all solutions of (*) \((Q_2)\) are oscillation in \( G \). The proof of theorem 2 is therefore completed.

3 Some Note of Several Oscillation Criteria

We may extend the results of the impulsive hyperbolic equations for \((2r+1)\) order case by using some definitions and some stating results in [3]. When \( r = 0 \) or \( r = 1 \) we will give out some results in [3]-[4] respectively, which is also new things for this direction.

In this section, let \( \Omega \) also be a bounded domain in \( \mathbb{R}^n \) with a piecewise smooth boundary \( \partial \Omega \), and \( PC(R_+, R_+) = \{ x(t) : R_+ \rightarrow R_+ \}, x(t) \) is piecewise continuous for \( t \in R_, t \neq t_k, x(t_k^+), x(t_k^-) \) exist and \( x(t_k) = x(t_k^+), k = 1, 2, \cdots \),
\[ \lim_{k \to \infty} t_k = \infty, 0 < t_1 < t_2 < \cdots < t_k < \infty, \text{ etc.} \]

We make it satisfy following conditions:

\( (H_1) \)
\[ a(t), a_i (t) \in PC(R_+, R_+), \lambda_i (t) \in PC^2 (R_+, R_+), (i = 1, 2, \cdots, m); \sigma (t), \rho_j (t) \in PC (R_+, R_+) \text{ and } \lim_{t \to \infty} \sigma (t) = \lim_{t \to \infty} \rho_j (t) = \infty, \text{ and } I : \Omega \times R_+ \times R \rightarrow R, f \in PC(G, R). \]

\( (H_2) \)
\[ c(x, t, \xi, \eta) \in PC(G \times R \times R, R), \quad c(x, t, \xi, \eta) \geq p(t) h(\xi) \text{ for all } (x, t, \xi, \eta) \in G \times R_+ \times R_+, t \neq t_k, \text{ where } p(t) \in PC(R_+, R_+) \text{ and that his continuous, positive and convex function in } R_+. \]
We assume that they are left continuous, at the moments of impulse , the following relations \( u(x, t^+_k) = u(x, t_k) \), and \( u(x, t^+_k) = u(x, t_k) + I(x, t_k, u(x, t_k)) \), are satisfied.

We consider the systems:

\[
\frac{\partial^2}{\partial t^2} [u + \sum_{j=1}^n \lambda_j(t)u(x, t - \tau_j)] = a(t)\Delta^{2r+1}u + \sum_{j=1}^k a_j(t)\Delta^{2r+1}u(x, \rho_j(t)) - C(x, t, u(x, t), u(x, \sigma(t)) + f(x, t),
\]

\((x, t) \in \Omega \times (0, \infty) = G, t \neq t_k.\)

\[
u(x, t^+_k) - u(x, t^-_k) = I(x, t, u), t = t_k, k = 1, 2, \ldots
\]

(4)

with boundary condition:

\[
\frac{\partial u}{\partial \gamma} = \psi, \frac{\partial \Delta u}{\partial \gamma} = \psi_1, \frac{\partial \Delta^2 u}{\partial \gamma} = \psi_2, \ldots, \frac{\partial \Delta^{2r} u}{\partial \gamma} = \psi_{2r} \text{ on } \partial \Omega \times R_+, t \neq t_k, \quad (B)
\]

**Theorem 3** Assume that \((H_1) - (H_2)\) hold, and satisfy \((A)\) for any function \(u \in PC(\Omega \times R_+, R_+)\) and constant \(\alpha_k > 0\) those

\[
\int_{\Omega} I(x, t_k, u(x, t_k))dx \leq \alpha_k \int_{\Omega} u(x, t_k)dx, k = 1, 2, \ldots
\]

hold.

If \(u(x, t)\) is a positive solution of the problem (4)-(B) in the domain \(\Omega \times [t_0, \infty)\) for some \(t_0 > 0\), then the impulsive differential inequalities of neutral type

\[
[W(t) = \sum_{i=1}^m \lambda_i(t)W(t - \tau_i)]^r + p(t)h(W(\sigma(t)) \leq H(t), \quad (x, t) \in \Omega \times [t_0, \infty), t \neq t_k, \quad
\]

\[
W(t^+_k) \leq (1 + \alpha_k)W(t_k), k = 1, 2, \ldots
\]

(5)

have an eventually positive solution

\[
W(t) = \frac{1}{|\Omega|} \int_{\Omega} u(x, t)dx
\]
where \( H(t) = \frac{1}{|Ω|} \left\{ \int_Ω [a(t)ψ_{2r}(x,t) + \sum_{j=1}^k a_j(t)ψ_{2r}(x,\rho_j(t))]ds + \int_Ω f(x,t)dx \right\}, \)

\( t \neq t_k \),

**Proof** Let \( u(x,t) \) be a positive solution of problem (4)-(B) in the domain \( Ω \times [\tau_0, +∞) \) for some \( \tau_0 > 0 \).

For \( t \neq t_k \), it follows from \((H_1)\) that there exists a \( t_1 \geq \tau_0 \) such that

\[ u(x,t - \tau_j) > 0, u(x,\rho_j(t)) > 0, u(x,\sigma(t)) > 0 \quad \text{in} \quad Ω \times [\tau_1, \infty), \quad i = 1,2,\cdots,m, \quad j = 1,2,\cdots,k. \]

Thus, we obtain that

\[
\frac{∂^2}{∂t^2} [u + \sum_{j=1}^m \lambda_j(t) u(x,t - \tau_j)] \leq a(t)Δ^{2r+1} u + \sum_{j=1}^k a_j(t)Δ^{2r+1} u(x,\rho_j(t)) - p(t)h(u(x,\sigma(t))) + f(x,t), \quad (x,t) ∈ Ω \times [\tau_1, \infty), \quad t \neq t_k. \quad (6)
\]

From condition (B), Green identity and Jensen’s inequality, it follows that

\[
\int_Ω Δ u dx = \int_Ω \frac{∂u}{∂γ} ds = \int_Ω ψ dy, \quad \int_Ω Δ^2 u dx = \int_Ω \frac{∂Δu}{∂γ} ds = \int_Ω ψ dy, \quad \cdots, \quad \int_Ω Δ^{2r+1} u dx = \int_Ω ψ_{2r} dy,
\]

\[
\int_Ω Δ u(x,\rho_j(t)) dx = \int_Ω \frac{∂}{∂\rho_j} u(x,\rho_j(t)) ds = \int_Ω ψ(x,\rho_j(t)) ds, \quad \text{and by similar}
\]

calculating this integration we have that

\[
\int_Ω Δ^{2r+1} u(x,\rho_j(t)) dx = \int_Ω ψ_{2r+1}(x,\rho_j(t)) ds.
\]

Therefore integrating (6) for \( x \) over \( Ω \), we obtain

\[
\frac{d^2}{dt^2} [\int_Ω u dx + \sum_{j=1}^m \lambda_j(t)\int_Ω u(x,t - \tau_j) dx]
\]
\[ \int_\Omega f(x,t)dx \leq a(t)\int_\Omega A_{2r+1}udx + \sum_{j=1}^{k} a_j(t)\int_\Omega A_{2r+1} u(x,\rho_j(t))dx - p(t)\int_\Omega h(u(x,\sigma(t)))dx + \int_\Omega f(x,t)dx \]

\[ \leq a(t)\int_\Omega \psi_{2r} ds + \sum_{j=1}^{k} a_j(t)\int_\Omega \psi_{2r} (x,\rho_j(t))ds - p(t)\int_\Omega h\left(\frac{1}{|\Omega|}\int_\Omega u(x,\sigma(t))dx\right) \]

\[ + \int_\Omega f(x,t)dx, \quad t \neq t_k, t \geq t_1. \]

where \(|\Omega| = \int_\Omega dx\). Set \( W(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx \), Thus we have

\[ \{W(t) + \sum_{j=1}^{m} \lambda_j(t)W(t - \tau_j)\}'' + p(t)h\{W(\sigma(t))\} \]

\[ \leq \frac{1}{|\Omega|}\left\{\int_\Omega [a(t)\psi_{2r}(x,t) + \sum_{j=1}^{k} a_j(t)\psi_{2r} (x,\rho_j(t))]ds + \int_\Omega f(x,t)dx\right\} \]

\[ = H(t), \quad (x,t) \in \Omega \times [t_1, \infty), t \neq t_k, \quad (7) \]

For \( t = t_k \), by (4) we have \((k = 1,2,\cdots)\)

\[ \int_\Omega (u(x,t_k) - u(x,t_k))\varphi(x)dx = \int_\Omega (\varphi(x),u(x,t_k))\varphi(x)dx \leq \alpha_k \int_\Omega u(x,t_k)\varphi(x)dx, \]

So \( \int_\Omega u(x,t_k)\varphi(x)dx \leq (1 + \alpha_k)\int_\Omega u(x,t)\varphi(x)dx, \quad (k = 1,2,\cdots) \quad (8) \)

Hence the inequalities (7)-(8) imply that the function \( W(t) \) is a positive solution of the impulsive differential inequality of neutral type in (4) for \( t \geq t_1 \). Therefore this ends the proof.

Remark. When \( r = 0 \) we get the theorem 2.3 in [5], and when \( r = 1 \) that is a sixth-order case.

Theorem 4 Assume that same as theorem 3 that \((H_1) - (H_2)\) and \((A)\) hold, and that

\[ (A') \quad c(x,t,\xi,\eta) = -c(x,t,\xi,\eta) \quad \text{for all} \quad (x,t,\xi,\eta) \in G \times R \times R, \quad t \neq t_k, \]
Oscillation theorems for systems of PDE

\[ I(x, t_k, u(x, t_k)) = -I(x, t_k, u(x, t_k)), \quad t = t_k, \quad (k = 1, 2, \cdots), \] and both the impulsive differential inequalities of neutral type (4) and

\[
[V(t) + \sum_{i=1}^{m} \lambda_i(t) V(t - \tau_i)][n] + (\lambda_0)^{2r+1} a(t) V(t) + (\lambda_0)^{2r+1} \sum_{j=1}^{k} a_j(t) V(\rho_j(t))
\]

\[ + p(t)h(V(o(t))) \leq -F(t), \quad t \neq t_k,
\]

\[ V(t^*_k) \leq (1 + \alpha_j) V(t^*_k), \quad k = 1, 2, \cdots \quad (8*) \]

have no eventually positive solution. Then there every nonzero solution of the problem (4)-(5) is Oscillation in the domain \( G = \Omega \times R_+ \).

Proof The proof is similar to the theorem 2 in [4], so we omit it.

Remark When \( r=1 \) we get the theorems 1-2 of [3], and When \( r=0 \) we get the theorem 2.2 of [5]. There is taking \( r = 2, 3, \cdots \), then now we have more results.

4 Some examples

**Example 1** We consider that system (5)-(5)’:

\[
\frac{\partial^6 u_1(x, t)}{\partial t^6} + \frac{\partial^5 u_1(x, t)}{\partial t^5} = (\Delta^3 + \Delta^2 + 4\Delta) u_1(x, t) + \frac{1}{2} \Delta u_1(x, t - \frac{3\pi}{2})
\]

\[ - 3u_1(x, t - 3\pi) - \frac{3}{2} u_2(x, t - 3\pi) \quad (9) \]

\[
\frac{\partial^6 u_2(x, t)}{\partial t^6} + \frac{\partial^5 u_2(x, t)}{\partial t^5} = (\Delta^3 + \Delta^2 + 4\Delta) u_2(x, t) + \frac{1}{2} \Delta u_2(x, t - \frac{3\pi}{2})
\]

\[ - \left(-\frac{3}{2}\right) u_1(x, t - \pi) - 3 u_2(x, t - \pi) \quad (9)’ \]

where \((x, t) \in (0, \pi) \times [0, \infty)\). The boundary value condition:

\[
\frac{\partial}{\partial x} u_i(0, t) = \frac{\partial}{\partial x} u_i(\pi, t) = 0, \quad \frac{\partial^2}{\partial x^2} u_i(0, t) = \frac{\partial^2}{\partial x^2} u_i(\pi, t) = 0, \quad t \geq 0, \quad i = 1, 2.
\]

Let \( n = 6, N = 1, m = 2, s = 1, a_i(t) = 4, a_{i1}(t) = \frac{1}{2}, \rho_i(t) = \frac{3\pi}{2}, \sigma(t) = \pi \),
\[
p_{11}(x,t) = 3, \quad p_{12}(x,t) = \frac{3}{2}, \quad p_{21}(x,t) = -\frac{3}{2}, \quad p_{22}(x,t) = 3; \quad a_2(t) = 4, \quad a_{21}(t) = \frac{1}{2},
\]
\[
\Omega = (0, \pi), \quad \text{and} \quad Q(t) = \frac{3}{2}.
\]

It satisfy all condition of theorem 1, then all solution of this system are oscillation on \((0, \pi) \times [0, \infty)\) (In fact, we have that \(u_1(x,t) = \cos x \sin t\), \(u_2(x,t) = \cos x \cos t\) are oscillation solution of the system (9)-(9)’).

References


[8] Chen Ning, Blow up of solution for a kind of six order hyperbolic and parabolic evolution systems, Applied Mathematical Science VoL.1.no.25(2007), 131-140.

Received: August, 2008