A Goodness of Fit Approach to Monotone Variance Residual Life Class of Life Distributions

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Abstract

Based on the goodness of fit approach, a new test is presented for testing exponentiality versus decreasing (increasing) variance remaining life distribution DVRL (IVRL). The percentiles of these test are tabulated for sample sizes $n=5(1)6(2)50$. It is shown that the proposed test is simple, has high relative efficiency for some commonly used alternatives and enjoys a good power. An example in medical science is considered as a practical application of the proposed test.

Keywords: DVRL (IVRL) class of life distribution, exponentiality, efficiency, asymptotic normality

1 Introduction

The variance residual life (VRL) distributions are useful in many areas including biometry, actuarial science and reliability. Let $T$ denote the life time of an equipment distribution function $F(x)$, survival function $\bar{F} = 1 - F$, mean life $\mu = \int_0^\infty \bar{F}(u)du$ and variance life $\sigma^2 = var(T)$ both assumed finite. The mean residual life (MRL) and the variance residual life (VRL) are defined as the following:

$$\mu(t) = E\{T - t|T \geq t\} = \frac{\int_t^\infty \bar{F}(u)du}{F(t)}, \quad t \geq 0, \quad (1.1)$$

and

$$\sigma^2(t) = var\{T - t|T \geq t\} = var\{T|T \geq t\}. \quad (1.2)$$
Consider \( E[U^2/t] = - \int_0^\infty u^2 d\overline{F}(u/t) \), integrating by parts, one have
\[
\sigma^2(t) = \frac{2}{F(t)} \int_0^\infty \int_y^\infty \overline{F}(x) dx dy - \mu^2(t),
\] (1.3)

let \( \nu(y) = \int_y^\infty \overline{F}(x) dx \) and \( \Gamma(t) = \int_t^\infty \nu(y) dy \), then (1.1) and (1.3) become as the following:
\[
\mu(t) = \frac{\nu(t)}{F(t)},
\] (1.4)

and
\[
\sigma^2(t) = \frac{2}{F(t)} \int_t^\infty \nu(y) dy = \frac{2\Gamma(t)}{F(t)} - \mu^2(t).\]
(1.5)

or
\[
\sigma^2(t) = \frac{2\overline{F}(t)\Gamma(t) - \mu^2(t)}{\overline{F}^2(t)}.\]
(1.6)

A distribution function \( F \) is said to be a decreasing (increasing ) variance residual life DVRL (IVRL) if \( \sigma^2(t) \) is nondecreasing (nondecreasing) function of \( t \) ( i.e. \( \frac{d\sigma^2(t)}{dt} \leq (\geq)0 \)). Differentiating (1.4) and (1.5) with respect to \( t \), we have
\[
\frac{d\mu(t)}{dt} = -1 + \nu(t)\mu(t). \]
(1.7)

\[
\frac{d\sigma^2(t)}{dt} = \frac{2f(t)\Gamma(t)}{\overline{F}^2(t)} - \frac{2\nu(t)}{F(t)} - 2\mu(t)\frac{d\mu(t)}{dt}. \]
(1.8)

Using (1.4) and (1.7) in (1.8), we obtain
\[
\frac{d\sigma^2(t)}{dt} = r(t)[\sigma^2(t) - \mu^2(t)],
\]
where \( r(t) = \frac{f(t)}{F(t)} \). Since \( r(t) \) is nonnegative for all \( t \), let us recall that \( \overline{F}(t) \) is DVRL (IVRL) if \( \sigma^2(t) \leq (\geq)\mu^2(t) \), by using (1.16) this implies that \( \overline{F}(t) \) is DVRL (IVRL) if
\[
2\overline{F}(t)\Gamma(t) \leq (\geq) 2\nu^2(t).
\]

Now, we have the following definition

**Definition (1.1):** A life distribution \( F \), with \( F(0) = 0 \) and its survival function \( \overline{F} \) is said to have DVRL (IVRL) class of life distributions if
\[
\overline{F}(t)\Gamma(t) \leq (\geq)\nu^2(t),
\] (1.9)

Launcer (1987), Gupta (1987), Gupta et al (1987), Kanjo (1996) and Gupta and Kirmani (2000) are studied characterization of this class and used it to
find better bounds on moments and survival function. The null distribution
for DVRL(IVRL) is the exponential. Thus we often encounter testing $H_0$ : A
life distribution is exponential versus $H_1 :$ It is DVRL (IVRL) and not expo-
nential. This testing problem was investigated by Kanwar and Madhu (1991),
Fango (1996) and recently by Abu-Youssef (2004, 2007). However in contrast
to goodness of fit problems, where the test statistics is based on a measure
of departure from $H_0$ that depends on both $H_0$ and $H_1$. Most tests of life
testing setting included those refereed above did not use the null distribution
in devising the test statistics, which resulted in test statistics that are often
difficult to work with require programming to evaluate.
Recently Ahmad et al. (2001), El-Bassiouny and El- Wasel (2003) and Abu-
Youssef (2007) were used a new methodology for testing by incorporating both
$H_0$ and $H_1$ in devising the test statistics for testing $H_0$ against the alternative
the life distribution is IFR, NBUC, HNBUE and DMRL classes of life dis-
tributions. They obtained very simple statistics that are not asymptotically
equivalent in distribution and efficiency to classical procedure but also better
in finite sample behaviors. Our goal in this paper is to use similar methodol-
ogy to obtain a very simple statistics for testing $H_0$ against $H_1$. The thread
that connects most work mentioned here is that a measure of departure from
$H_0$, which is strictly positive under $H_1$ and is zero under $H_0$. Then, a sample
version of this measure is used as test statistics and its properties are stud-
ied. In section 2, we propose a test statistic, based on the goodness of fit
approach, for testing $H_0$: $F$ is exponential against $H_1$: $F$ is DVRL (IVRL)
and not exponential. We then present Monte Carlo null distribution critical
points for sample sizes $n = 5(1)6(2)50$. In section 3 we calculate the efficiency
of the test statistic for some common alternatives and compared them to other
procedures. In section 4 we give simulated values of the power estimates of the
test. Finally an application in medical science was introduced in section 5.

2 Testing DVRL (IVRL) class of life distribu-
tion

The test presented have depends on a sample $X_1, X_2, \ldots, X_n$ from a pop-
ulation with distribution $F$. We wish to test the null hypothesis $H_0 : F$ is
exponential with mean $\mu$ against $H_1 : F$ is DVRL (IVRL) class of life distri-
bution and not exponential, using the inequality (1.9), one used the following
as a measure of departure from $H_0$ in favor of $H_1$:

$$\delta_v = \int_0^\infty [\nu^2(t) - \bar{F}(t)\Gamma(t)]dt. \tag{2.1}$$

But

$$\int_0^\infty \nu^2(y)dy = \int_0^\infty x^2
$$

and

$$\int_0^\infty F(x)\Gamma(x)dx = \int_0^\infty x\Gamma(x)dF(x) + \frac{1}{2}\int_0^\infty x^2\nu(x)dF(x) + \frac{1}{3}x^3F(x)dF(x). \tag{2.3}$$

Then, from (2.2) and (2.3), the measure in (2.1) becomes as the following:

$$\delta_v = \int_0^\infty \left[ \frac{x^2}{2}\nu(t) + \frac{x^3}{3}\bar{F}(t) - x\Gamma(t) \right]dF(t). \tag{2.4}$$

Note that under $H_0: \delta_v = 0$, while under $H_1: \delta_v > (\sim)0$

Denote $X_1, X_2, \ldots, X_n$ be the corresponding ordered sample and if $F_n = F_{X_1, X_2, \ldots, X_n}$ is the empirical distribution function, then $\bar{F}_n = \frac{1}{n}\sum_{j=1}^n I(X_j > x)$ is empirical survival function, where $i = 1, 2, \ldots, n$. and the empirical functions of $\delta_v$ and $\Gamma_n(x)$ are $\hat{\nu}_n(x) = \frac{1}{n}\sum_{j=1}^n (X_j - x)I(X_j > x)$ and $\hat{\Gamma}_n(x) = \frac{1}{2n}\sum_{j=1}^n (X_j - x)^2I(X_j > x)$ respectively; whereas

$$I(X_j > x) = \begin{cases} 1 & X_j > x \\ 0 & \text{otherwise} \end{cases}$$

In a similar fashion, if $F_0$ denote the exponential distribution, we can take in place of (2.1) or (2.4) the following measure of departure from $H_0$

$$\delta_{v_1} = \int_0^\infty \left[ \frac{x^2}{2}\nu(t) + \frac{x^3}{3}\bar{F}(t) - x\Gamma(t) \right]dF_0(t), \tag{2.5}$$

for testing the hypothesis that $H_0$: $F$ is exponential versus $H_1$: $F$ is DVRL (IVRL) class of life distribution and not exponential. With out loss of generality we take $\mu = 1$ and thus $F_0(x) = 1 - e^{-x}$. In order to derive an expression for $\delta_{v_1n}$, we need the following theorem.

**Theorem 2.1.** Let $T$ be a variable with distribution function $F$. Then

$$\delta_{v_1} = -4 + E[3X - \frac{1}{2}X^2 + e^{-X}(4 + X - \frac{1}{2}X^2 - \frac{1}{3}X^3)]. \tag{2.6}$$
Proof. Note that \( \delta_{v1} \) in (2.5) be written as the following:

\[
\delta_{v1} = \int_{0}^{\infty} \frac{x^2}{2} \nu(x) e^{-x} dx + \int_{0}^{\infty} \frac{x^3}{3} \frac{\Phi(x)}{\Gamma(x)} e^{-x} dx - \int_{0}^{\infty} x \Gamma(x) e^{-x} dx
= I1 + I2 - I3
\]

(2.7)

where \( I1 = \int_{0}^{\infty} \frac{x^2}{2} \nu(x) e^{-x} dx \), \( I2 = \int_{0}^{\infty} \frac{x^3}{3} \frac{\Phi(x)}{\Gamma(x)} dx \) and \( I3 = \int_{0}^{\infty} x \Gamma(x) dx \)

But

\[
I1 = \int_{0}^{\infty} E(X - x) I(X > x) \frac{x^2}{2} e^{-x} dx
= E \int_{0}^{X} (X - x) \frac{x^2}{2} e^{-x} dx
= -3 + E[X + (3 + 2X + \frac{1}{2}X^2) e^{-X}],
\]

(2.8)

\[
I2 = \int_{0}^{\infty} \frac{x^3}{3} E I(X > x) e^{-x} dx
= E \int_{0}^{X} \frac{x^3}{3} e^{-x} dx
= 2 + E[(-2 - 2X - X^2 - \frac{1}{3}X^3) e^{-X}],
\]

(2.9)

and

\[
I3 = \frac{1}{2} \int_{0}^{\infty} x E(X - x)^2 I(X > x) e^{-x} dx
= \frac{1}{2} E \int_{0}^{X} x(X - x)^2 e^{-x} dx
= 3 + E[-2X + \frac{1}{2}X^2 - (3 + X) e^{-X}].
\]

(2.10)

Using (2.8), (2.9) and (2.10) in (2.7), we get the result.

Note that: \( \delta_{v1} = 0 \) under \( H_0 \), while it is positive under \( H_1 \). Thus based on a random sample \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \) from a distribution \( F \). We wish to test \( H_0 \) against \( H_1 \), we may be testing on its estimate. A direct empirical estimate of \( \delta_{v1} \) is

\[
\hat{\delta}_{v1n} = -4 + \frac{1}{n} \sum_{i=1}^{n} \left\{ 3X_i - \frac{1}{2}X_i^2 + e^{-X_i} (4 + X_i - \frac{1}{2}X_i^2 - \frac{1}{3}X_i^3) \right\}.
\]

(2.11)

To make the test scale invariant, we take

\[
\hat{\Delta}_{v1n} = \frac{\hat{\delta}_{v1n}}{X_n}
\]

(2.12)
Theorem 3.1. As \( n \to \infty \), \( \sqrt{n}(\hat{\Delta}_{v_1n} - \Delta_{v_1n}) \) is asymptotically normal with mean 0 and variance \( \sigma^2 \) where \( \sigma^2 \) is given in (2.13). Under \( H_0: \sigma_0^2 = 0.27869 \).

Proof. Since \( \hat{\Delta}_{V_1n} \) and \( \frac{\hat{v}_{1n}}{\mu^3} \) have the same limiting distribution, we use \( \sqrt{n}(\hat{\delta}_{v_1n} - \delta_{v_1n}) \). Noting that \( \hat{\delta}_{v_1n} \) is just an average, it is straightforward by using the central limit theorem the result follows. For the variance

\[
\sigma^2 = E[-4 + 3X - \frac{1}{2}X^2 + e^{-4X}(4 + X - \frac{1}{2}X^2 - \frac{1}{2}X^3)]^2. \tag{2.13}
\]

Under \( H_0, \Delta_{v1} = 0 \) and

\[
\sigma_0^2 = \int_{0}^{\infty} [-4 + 3X - \frac{1}{2}X^2 + e^{-X}(4 + X - \frac{1}{2}X^2 - \frac{1}{2}X^3)]^2 e^{-x} \, dx = 0.27869.
\]

Then the theorem is proved.

3 Monte carlo null distribution critical points for \( \hat{\Delta}_{F_n} \) test

In practice, simulated percentiles for small samples are commonly used by applied statisticians and reliability analyst. we have simulated the upper percentile points for 90%, 95%, 99%. Table (3.1) gives these percentile points of statistic \( \hat{\Delta}_{v_1n} \) in (2.12) and the calculations are based on 5000 simulated samples of sizes \( n = 5(1)6(2)50 \). The percentile values change slowly as \( n \) increase.

Table 3.1 Critical Values of \( \hat{\Delta}_{v_1n} \)
To use the above test, calculate $\sqrt{n} \hat{\Delta}_{v_1n}/\sigma_0^2$ and reject $H_0$ if this exceeds the normal variate value $Z_{1-\alpha}$.

### 4 Asymptotic relative efficiency (ARE)

We compare our test $\hat{\Delta}_{v_1n}$ to tests $\hat{\Delta}_{vn}$ and $\hat{\Delta}_{kv_n}$ presented by Abu-Youssef (2004, 2007) for DVR classes of life distributions. The comparisons are achieved by using Pitman asymptotic relative efficiency (PARE), which is defined as follows:

Let $T_{1n}$ and $T_{2n}$ be two statistics for testing $H_0$: $F_{\theta} \{ F_{\theta_n} \}$, $\theta_n = \theta + c\sqrt{n}$ with $c$ an arbitrary constant, then PARE of $T_{1n}$ relative to $T_{2n}$ is defined by

$$e(T_{1n}, T_{2n}) = \frac{\mu_1(\theta_0)/\sigma_1(\theta_0)}{\mu_2(\theta_0)/\sigma_2(\theta_0)}$$
where \( \mu_i(\theta_o) = \lim_{n \to \infty} \frac{\partial}{\partial \theta} E(T_{in}) \rvert_{\theta = \theta_o} \) and \( \sigma^2_i(\theta_o) = \lim_{n \to \infty} \text{Var} E(T_{in}) \), \( i = 1, 2 \). Two of the most commonly used alternatives (cf. Hollander and Proschan (1972)) are:

(i) Linear failure rate family : \( \bar{F}_{1\theta} = e^{-x - \frac{\theta x^2}{2}}, \ x > 0, \theta > 0 \)
(ii) Weibull family : \( \bar{F}_{2\theta} = e^{-x^\theta}, \ x \geq 0, \theta > 0 \)

The null hypothesis is at \( \theta = 0 \) for linear failure rate and \( \theta = 1 \) for Weibull family. Direct calculations of PAE of \( \hat{\Delta}_{v1n}, \hat{\Delta}_{vn} \) and \( \hat{\Delta}_{kv_n} \) are summarized in Table (4.1).

**Table 4.1** PAE of \( \hat{\Delta}_{v1n}, \hat{\Delta}_{vn} \) and \( \hat{\Delta}_{kv_n} \)

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( \Delta_{v1n} )</th>
<th>( \Delta_{vn} )</th>
<th>( \Delta_{kv_n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 ) (Linear failure rate)</td>
<td>1.54</td>
<td>0.91</td>
<td>0.94</td>
</tr>
<tr>
<td>( F_2 ) (Weibull)</td>
<td>0.74</td>
<td>1.83</td>
<td>1.89</td>
</tr>
</tbody>
</table>

The efficiencies in Table (4.1) show clearly our statistic \( \hat{\Delta}_{v1n} \) performs better than \( \hat{\Delta}_{vn} \) for \( F_1 \), also it performs better than \( \hat{\Delta}_{kv_n} \) for \( F_1 \).

In table 4.2 we give PARE’s of \( \hat{\Delta}_{v1n} \) with respect to \( \hat{\Delta}_{vn} \) and \( \hat{\Delta}_{kv_n} \) whose PAE are mentioned in table 4.1.

**Table 4.2** PARE of \( \hat{\Delta}_{v1n} \) with respect to \( \hat{\Delta}_{vn} \) and \( \hat{\Delta}_{kv_n} \)

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( e_{F_1}(\hat{\Delta}<em>{v1n},\hat{\Delta}</em>{vn}) )</th>
<th>( e_{F_1}(\hat{\Delta}<em>{v1n},\hat{\Delta}</em>{kv_n}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 ) (Linear failure rate)</td>
<td>1.69</td>
<td>1.64</td>
</tr>
<tr>
<td>( F_2 ) (Weibull)</td>
<td>0.40</td>
<td>0.39</td>
</tr>
</tbody>
</table>

It is clear from Table 4.2 that the statistic \( \hat{\Delta}_{v1n} \) performs well for \( F_1 \) and it is more efficient. Finally, the power of the test statistics \( \hat{\Delta}_{v1n} \) is considered for 95% percentiles in Table 4.3 for two of the most commonly used alternatives [see Hollander and Proschan (1975)], they are:

(i) Linear failure rate : \( \bar{F}_{\theta} = e^{-x - \frac{\theta x^2}{2}}, \ x > 0, \theta > 0 \)
(ii) Weibull family : \( \bar{F}_{3\theta} = e^{-x^\theta}, \ x \geq 0, \theta > 0 \)

These distributions are reduced to exponential distribution for appropriate values of \( \theta \).

**Table 4.3** Power Estimate of \( \hat{\Delta}_{v1n} \)
5 Numerical Examples

Consider the data in Susarla and Van Ryzin (1978). These data represent 81 patients of melanoma. Of them 46 represent whole life time (non-censored data) and the ordered values are: 13, 14, 19, 19, 20, 21, 23, 23, 25, 26, 26, 27, 27, 31, 32, 34, 34, 37, 38, 38, 40, 46, 50, 53, 54, 57, 58, 59, 60, 65, 65, 66, 70, 85, 90, 98, 102, 103, 110, 118, 124, 130, 136, 138, 141, 234.

Using equation (2.12), the value of test statistics, based on the above data is $\hat{\Delta}_{v_1 n} = -0.0114$. This value leads to $H_0$ is not rejected at the significance level $\alpha = 0.05$. See Table (3.1). Therefore the data has not DVR Property.

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References


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