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Ky Fan Inequality for Set-Valued Maps

Xiaojun Yu

School of Mathematics and Statistics
Guizhou College of Finance and Economics
Guiyang 550004, P. R. China
xjyu-myu@163.com

Abstract

In this paper, employing the Ky Fan section theorem,we give a class of Ky Fan inequality for set-valued maps,usually it takes Ky Fan inequality ,vector equilibrium problems as its special cases.

Keywords: Set-valued map; lower C-semicontinuous; C-concave; Ky Fan inequality

1 Introduction

In [1] Ky Fan obtained the famous Ky Fan inequality as following:

Let X be a nonempty compact convex subset of a Hausdorff topological vector space, $\Phi : X \times X \to R$ such that:
(1) for each fixed $y \in X, \Phi(\cdot, y)$ is lower semicontinuous;
(2) for each fixed $x \in X, \Phi(x, \cdot)$ is quasi-concave;
(3) for all $x \in X, \Phi(x, x) \leq 0$.
Then, there exists $x^* \in X$ such that $\Phi(x^*, y) \leq 0$ for all $y \in X$.

Ky Fan inequality is a very important theorem in nonlinear analysis. Ky Fan inequality has become important topics in optimal theory and Game theory ,see [3, 6] and references therein. In [4], Tan,Yu and Yuan define the Ky Fan's point,and they proved the generic stability of Ky Fan's points. Yu and Xiang[7] proposed the essential components of Ky Fan's point. They proved that under some conditions the Ky Fan's points have at least one essential component. Besides ,they proved that for any $n-$ person noncooperative game, there exist at least one essential components of the set of its Nash equilibrium points. Yang and Yu[5] give a generalization of Ky Fan's inequality to vector-valued function. They proved that , for every vector-valued function(satisfying some continuity and convexity condition), there exists at least one essential components of the set of its Ky Fan's points. As application, they show that , for every multiobject game, there exists at least one essential components of the set of weakly Pareto-Nash equilibrium points.

In this paper, employing the Ky Fan section theorem,we give a class of Ky Fan inequality for set-valued maps,usually it takes Ky Fan inequality ,vector equilibrium problems as its special cases.

2 Preliminaries

Definition 1. Let Y be a real topological vector space. A subset $C \subset Y$ is said to be a cone if for all $c \in C$ and for all $t \in [0, +\infty)$ such that $tc \in C$. A cone C is convex if and only if $C + C = C$, and pointed if and only if $C \cap (-C) = \{\theta\}$,where θ denotes the zero elements of Y.

Definition 2. Let E, H be two Hausdorff topological vector spaces,C be a nonempty closed,convex and pointed cone of H. X be a subset of E. and $F : X \to 2^H$ be a set-valued map, F is said to be lower C-semicontinuous at $x_0 \in X$, if for any open set $V \subset H, V \cap F(x_0) \neq \emptyset$, there exists a neighborhood U of x_0 in E, for all $x \in U$,$F(x) \cap (V + C) \neq \emptyset$. and lower C-semicontinuous on X if it is lower C-semicontinuous at every point of X.

Remark 1: If F is vector-valued function ,then the lower C-semicontinuous is coincide with C-continuity.

Definition 3. Let X be a nonempty closed,convex subset in topological vector space,$f : X \to H$ be a vector valued function.f is said to be C-concave,if for any $x_1, x_2 \in X$, and any $\lambda \in [0, 1]$ such that

$$f(\lambda x_1 + (1 - \lambda)x_2) - (\lambda f(x_1) + (1 - \lambda)f(x_2)) \in C$$

and C-convex if $-f$ is C-concave.

Definition 4. Let X, Y be two Hausdorff topological vector spaces,C be a nonempty closed,convex,pointed cone of Y with $intC \neq \emptyset$,D be a nonempty convex subset of X and $F : D \to 2^Y$ be a set-valued map.F is said to be C-concave if for each $x_1, x_2 \in D$, and each $t \in [0, 1]$ such that

$$F(tx_1 + (1 - t)x_2) \subset tF(x_1) + (1 - t)F(x_2) + C$$

and C-convex if $-F$ is C-concave.

Remark 2: If F is vector-valued function ,then definition 4 is coincide with definition 3.

Lemma 1. (See lemma1.1 [5]) Let H be a Banach space with a closed,convex and pointed cone C with $intC \neq \emptyset$. Then we have $intC + C \subset intC$.

Lemma 2. Let E, H be two Hausdorff topological vector spaces,C be a nonempty closed,convex and pointed cone of H with $intC \neq \emptyset$. X be a nonempty convex subset of E. and $F : X \to 2^H$ be a set-valued map, and F is lower C-semicontinuous on X. Then the set $A = \{x \in X : F(x) \subset H \backslash intC\}$ is closed in X.

Proof. Let $\{x_\alpha\}$ be a net in X with $x_\alpha \to x$. Supposed $x \notin A$, then $F(x) \not\subset H \backslash intC$. i.e., there exists $y \in F(x)$ such that $y \in intC$. For $intC$ be a neighborhood of y and F is lower C-semicontinuous . Then, there exists a neighborhood U of x, for all $z \in U \cap X$ such that $F(z) \cap (intC + C) = F(z) \cap intC \neq \emptyset$ (1). For $x_\alpha \to x$,then there exists a β such that for all $\alpha \geq \beta, x_\alpha \in U \cap X$. Hence, for (1) can obtain that $F(x_\alpha) \cap intC \neq \emptyset$, i.e.,$x_\alpha \notin A$. which is contradiction. Therefore,the set $A = \{x \in X : F(x) \subset H \backslash intC\}$ is closed in X. The proof is complete.

Lemma 3. (Ky Fan section theorem,see[2]) Let X be a nonempty compact convex subset of a Hausdorff vector space and A be a subset of $X \times X$ such that:
(1) for each $y \in X$, the set $\{x \in X : (x, y) \in A\}$ is closed in X;
(2) for each $x \in X$, the set $\{y \in X : (x, y) \notin A\}$ is convex or empty;
(3) for each $x \in X, (x, x) \in A$.
Then there exists a point $x_0 \in X$ such that $\{x_0\} \times X \subset A$.

Let Y, H be two Hausdorff vector spaces,X be a nonempty subset of H,$\Phi : X \times X \to 2^Y, C : X \to 2^Y$ are set-valued maps and $C(x)$ be a nonempty closed,convex,pointed cone of Y with $intC(x) \neq \emptyset$.

The Ky Fan inequality of set-valued map is: find $x^* \in X$ such that for all $y \in X, \Phi(x^*, y) \cap intC(x^*) = \emptyset$. Then, x^* is said to be a set-valued Ky Fan's point of Φ.

Remark 3: (1) If set-valued map $C : X \to 2^Y$ such that for any $x \in X, C(x)$ be closed,convex cone of Y with $intC(x) \neq \emptyset$, and $\Phi : X \times X \to Y$ be a vector-valued function such that for all $x \in X, \Phi(x, x) \notin intC(x)$. Then the vector equilibrium problem is :find $x^* \in X$ such that $\Phi(x^*, y) \notin intC(x^*), \forall y \in X$.

Then if Φ is a vector-valued function,the set-valued Ky Fan's points are coincide with the vector equilibrium problem.

(2) If $Y = R, C = [0, +\infty)$, then Ky Fan's points of a set-valued map reduce to the Ky Fan's points of a real-valued function,defined by Tan,Yu and Yuan in [4].

3 Main results

Theorem 1. Let X be a nonempty compact convex subset of Hausdorff vector space E, Y be a Hausdorff vector space and $\Phi : X \times X \to 2^Y$ be a set-valued map with nonempty compact convex values. C be a nonempty closed,convex,pointed cone of Y with $intC \neq \emptyset$ such that:
(1)for each fixed $y \in X, \Phi(\cdot, y)$ is lower C-semicontinuous;
(2) for each fixed $x \in X, \Phi(x, \cdot)$ is C-concave;
(3) for each$x \in X$, such that $z \notin intC$ for all $z \in \Phi(x, x)$.
Then, there exists $x^* \in X$ such that $\Phi(x^*, y) \cap intC = \emptyset$ for all $y \in X$.

Proof. Let $A = \{(x, y) \in X \times X \, \forall z \in \Phi(x, y), z \notin intC\}$.

By condition (3), then for each $x \in X, (x, x) \in A$.

For each $y \in X$, the set A_y=$\{x \in X : (x, y) \in A\} = \{x \in X : \forall z \in \Phi(x, y), z \notin intC\} = \{x \in X : \forall z \in \Phi(x, y), z \in Y \backslash intC\} = \{x \in X : \Phi(x, y) \subset Y \backslash intC\}$.Then by lemma 2, it can obtain that the set A_y is closed.

For each $x \in X, A_x = \{y \in X : (x, y) \notin A\} = \{y \in X : \exists z \in \Phi(x, y), z \in intC\}$. Now, we prove that A_x is convex. Supposed that A_x is not a convex set. Then for each $y_1, y_2 \in A_x$, there exists $z_1 \in \Phi(x, y_1), z_1 \in intC, z_2 \in \Phi(x, y_2), z_2 \in intC$ such that for each $t \in [0, 1], z \notin intC$ for all $z \in \Phi(x, ty_1 + (1-t)y_2)$.Since $intC$ is convex, then there exists $z^* = tz_1+(1-t)z_2 \in intC$.For $\Phi(x, \cdot)$ is C-concave, then $\Phi(x, ty_1 + (1-t)y_2) \subset t\Phi(x, y_1) + (1-t)\Phi(x, y_2) + C$.For $tz_1 + (1-t)z_2 \in t\Phi(x, y_1) + (1-t)\Phi(x, y_2)$.Then, there exists $z \in \Phi(x, ty_1 + (1-t)y_2)$ such that $z \in tz_1 + (1-t)z_2 + C \subset intC + C \subset intC$ (by lemma 1). Which is contradiction. Then A_x is convex.

By lemma 3,then there exists $x^* \in X$ such that $\{x^*\} \times X \subset A$.i.e., $\Phi(x^*, y) \cap intC = \emptyset$ for all $y \in X$.The proof is complete.

Remark 4: If X is not a compact set,we can't use theorem 1,but the result may be exist.

For an example: Let $X = (-\infty, -2], Y = R, C = [0, +\infty)$. For each $x, y \in X$, let $\Phi(x, y) = 3x(x - y)$.Then, it can easily obtain that beside the compact condition, the other conditions of theorem 1 are satisfy,so we can't use the theorem 1.But there exists $x^* = -2 \in X$ such that:

$$\Phi(x^*, y) = -6(-2 - y) \leq 0, \forall y \in X$$

i.e., $\Phi(x^*, y) \cap intC = \emptyset, \forall y \in X$.

For theorem 1, we can obtain the existence theorem of vector equilibrium problem.

Corollary 1: Let X be a nonempty compact convex subset of a Hausdorff vector space E, and $\Phi : X \times X \to H$ such that:

(1) for each fixed $y \in X, \Phi(\cdot, y)$ is C-continuous;
(2) for each fixed $x \in X, \Phi(x, \cdot)$ is C-concave;
(3) for each $x \in X, \Phi(x, x) \notin intC$.
Then, there exists $x^* \in X$ such that $\Phi(x^*, y) \notin intC$ for all $y \in X$.

Remark 5: Corollary 1 is the Theorem 1.1 in [5].

In Theorem 1, let $Y = R, C = [0, +\infty)$, we can obtain the generalized Ky Fan inequality as following:

Corollary 2: Let X be a nonempty compact convex subset of a Hausdorff vector space E, and $\Phi : X \times X \to R$ such that:
(1) for each fixed $y \in X, \Phi(\cdot, y)$ is lower semicontinuous;
(2) for each fixed $x \in X$, the set$\{y \in X : \Phi(x, y) > 0\}$ is convex;
(3) for each $x \in X, \Phi(x, x) \leq 0$.
Then, there exists $x^* \in X$ such that $\Phi(x^*, y) \leq 0$ for all $y \in X$.

By corollary 2, it is easily to obtain the Ky Fan inequality in [1].

Corollary 3: (Ky Fan inequality,see Theorem 1 in [1])

Let X be a nonempty compact convex subset of a Hausdorff topological vector space, $\Phi : X \times X \to R$ such that:
(1) for each fixed $y \in X, \Phi(\cdot, y)$ is lower semicontinuous;
(2) for each fixed $x \in X, \Phi(x, \cdot)$ is quasi-concave;
(3) for all $x \in X, \Phi(x, x) \leq 0$.
Then, there exists $x^* \in X$ such that $\Phi(x^*, y) \leq 0$ for all $y \in X$.

Remark 6: There exist example show that Φ satisfy the condition (2) of corollary,but don't satisfy the condition (2) of corollary 3. Example:

Let $R = (-\infty, +\infty)$, for each $x \in R$,let $\Phi(x) = x^2 + 1$.Then, the set

$$\{x \in R : \Phi(x) > 0\} = \{x \in R = x^2 + 1 > 0\} = R$$

is a convex set.But, the function Φ is not a quasi-concave function in R.

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