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A Note on the Pell Equation $x^2 - Dy^2 = \pm 4$

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Abstract

Let D be any square-free positive integer. In this note we give a proof of two conjectures in [1] on the recurrence relations of integer solutions of the Pell equation $x^2 - Dy^2 = \pm 4$.

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1. Introduction

Let D be any square-free positive integer. In [1], A. Tekcan considered the Pell equation $x^2 - Dy^2 = \pm 4$ and proved some interesting results. He also proposed two conjectures on the recurrence relations of integer (actually positive) solutions in terms of the fundamental solution of the equation.

The aim of this note is just to show that these conjectures hold and follow immediately from the main results shown in [1], which, together with the conjectures, we record as follows.

Lemma 1.1. Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = 4$, and let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1}$$

for $n \geq 1$. Then all the positive integer solutions of the Pell equation $x^2 - Dy^2 = 4$ are given by $(x_n, y_n) = (\frac{u_n}{2^{n-1}}, \frac{v_n}{2^{n-1}})$.

Lemma 1.2. Let (x_1, y_1) be the fundamental solution of the negative Pell equation $x^2 - Dy^2 = -4$, and let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

for $n \geq 0$. Then all the positive integer solutions of the Pell equation $x^2 - Dy^2 = -4$ are given by $(x_{2n+1}, y_{2n+1}) = (\frac{u_{2n+1}}{2^{2n}}, \frac{v_{2n+1}}{2^{2n}})$.

Conjecture 1.3. Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = 4$, then (x_n, y_n) satisfy the following recurrence relations

$$\begin{aligned} x_n &= (x_1 - 1)(x_{n-1} + x_{n-2}) - x_{n-3} \\ y_n &= (x_1 - 1)(y_{n-1} + y_{n-2}) - y_{n-3} \end{aligned}$$

for $n \geq 4$.

Conjecture 1.4. Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = -4$, then (x_{2n+1}, y_{2n+1}) satisfy the following recurrence relations

$$\begin{aligned} x_{2n+1} &= (x_1^2 + 1)(x_{2n-1} + x_{2n-3}) - x_{2n-5} \\ y_{2n+1} &= (x_1^2 + 1)(y_{2n-1} + y_{2n-3}) - y_{2n-5} \end{aligned}$$

for $n \geq 3$.

2. Proof of conjectures

Proof of Conjecture 1.3. Since $(x_n, y_n) = (\frac{u_n}{2^{n-1}}, \frac{v_n}{2^{n-1}})$, we need only to show that

$$\begin{aligned} \begin{pmatrix} u_n \\ v_n \end{pmatrix} &= \begin{pmatrix} 2(x_1 - 1)(u_{n-1} + 2u_{n-2}) - 8u_{n-3} \\ 2(x_1 - 1)(v_{n-1} + 2v_{n-2}) - 8v_{n-3} \end{pmatrix} \\ &= 2(x_1 - 1)\left(\begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix} + 2\begin{pmatrix} u_{n-1} \\ v_{n-2} \end{pmatrix}\right) - 8\begin{pmatrix} u_{n-3} \\ v_{n-3} \end{pmatrix} \end{aligned}$$

In view of (1), it suffices to show

$$\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(2(x_1 - 1)\left(\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^2 + 2\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}\right) - 8\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

It is easy to see that

$$\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^2 = \begin{pmatrix} x_1^2 + Dy_1^2 & 2Dx_1y_1 \\ 2x_1y_1 & x_1^2 + Dy_1^2 \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^3 = \begin{pmatrix} x_1^3 + 3Dx_1y_1^2 & 3Dx_1^2y_1 + D^2y_1^2 \\ 3x_1^2y_1 + Dy_1^3 & x_1^3 + 3Dx_1y_1^2 \end{pmatrix}$$

Using $x_1^2 - Dy_1^2 = 4$, we have

$$\begin{aligned}
&\left(2(x_1-1)\left(\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^2 + 2\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}\right) - 8\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= 2(x_1-1)\left(\begin{pmatrix} x_1^2 + Dy_1^2 \\ 2x_1y_1 \end{pmatrix} + 2\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) - 8\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 2(x_1-1)(x_1^2 + Dy_1^2 + 2x_1) - 8 \\ 2(x_1-1)(2x_1y_1 + 2y_1) \end{pmatrix} \\
&= \begin{pmatrix} 2(x_1-1)(x_1^2 + (x_1^2 - 4) + 2x_1) - 8 \\ 2(x_1-1)(2x_1y_1 + 2y_1) \end{pmatrix} \\
&= \begin{pmatrix} 4(x_1-1)(x_1^2 + x_1 - 2) - 8 \\ 2(x_1-1)(2x_1y_1 + 2y_1) \end{pmatrix} \\
&= \begin{pmatrix} 4x_1(x_1^2 - 3) \\ 4(x_1^2 - 1)y_1 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} x_1^3 + 3Dx_1y_1^2 \\ 3x_1^2y_1 + Dy_1^3 \end{pmatrix} = \begin{pmatrix} x_1^3 + 3x_1(x_1^2 - 4) \\ 3x_1^2y_1 + y_1(x_1^2 - 4) \end{pmatrix} \\
&= \begin{pmatrix} 4x_1(x_1^2 - 3) \\ 4(x_1^2 - 1)y_1 \end{pmatrix}
\end{aligned}$$

This completes the proof. □

Proof of Conjecture 1.4. Since $(x_{2n+1}, y_{2n+1}) = (\frac{u_{2n+1}}{2^{2n}}, \frac{v_{2n+1}}{2^{2n}})$, we need only to show that

$$\begin{aligned}
\begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix} &= \begin{pmatrix} (x_1^2+1)(u_{2n-1} + u_{2n-3}) - u_{2n-5} \\ (x_1^2+1)(v_{2n-1} + v_{2n-3}) - v_{2n-5} \end{pmatrix} \\
&= (x_1^2+1)\left(\begin{pmatrix} u_{2n-1} \\ v_{2n-1} \end{pmatrix} + \begin{pmatrix} u_{2n-3} \\ v_{2n-3} \end{pmatrix}\right) - \begin{pmatrix} u_{2n-5} \\ v_{2n-5} \end{pmatrix}
\end{aligned}$$

In view of (2), it suffices to show

$$\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^6 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(4(x_1^2+1)\left(\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^4 + 4\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^2\right) - 64\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

It is easy to see that

$$\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^2 = \begin{pmatrix} x_1^2 + Dy_1^2 & 2Dx_1y_1 \\ 2x_1y_1 & x_1^2 + Dy_1^2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^4 = \begin{pmatrix} (x_1^2 + Dy_1^2)^2 + 4Dx_1^2y_1^2 & 4Dx_1y_1(x_1^2 + Dy_1^2) \\ 4x_1y_1(x_1^2 + Dy_1^2) & (x_1^2 + Dy_1^2)^2 + 4Dx_1^2y_1^2 \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^6 = \begin{pmatrix} (x_1^2+Dy_1^2)^3+12Dx_1^2y_1^2(x_1^2+Dy_1^2) & 6Dx_1y_1(x_1^2+Dy_1^2)^2+8D^2x_1^3y_1^3 \\ 8Dx_1^3y_1^3+6x_1y_1(x_1^2+Dy_1^2)^2 & (x_1^2+Dy_1^2)^3+12Dx_1^2y_1^2(x_1+Dy_1^2) \end{pmatrix}$$

Using $x_1^2 - Dy_1^2 = -4$, we have

$$\begin{aligned}
&\left(4(x_1^2+1)\left(\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^4 + 4\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^2\right) - 64\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= 4(x_1^2+1)\left(\begin{pmatrix} (x_1^2+Dy_1^2)^2+4Dx_1^2y_1^2 \\ 4x_1y_1(x_1^2+Dy_1^2) \end{pmatrix} + 4\begin{pmatrix} x_1^2+Dy_1^2 \\ 2x_1y_1 \end{pmatrix}\right) - 64\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 4(x_1^2+1)((x_1^2+Dy_1^2)^2+4Dx_1^2y_1^2+4(x_1^2+Dy_1^2)-64) \\ 4(x_1^2+1)(4x_1y_1(x_1^2+Dy_1^2)+4\cdot 2x_1y_1) \end{pmatrix} \\
&= \begin{pmatrix} 4(x_1^2+1)((x_1^2+x_1^2+4)^2+4x_1^2(x_1^2+4)+4(x_1^2+x_1^2+4))-64 \\ 4(x_1^2+1)(4x_1y_1(x_1^2+x_1^2+4)+8x_1y_1) \end{pmatrix} \\
&= \begin{pmatrix} 32((x_1^2+1)^2(x_1^2+4)-2) \\ 32x_1y_1(x_1^2+1)(x_1^2+3) \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^6 \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} (x_1^2+Dy_1^2)^3+12Dx_1^2y_1^2(x_1^2+Dy_1^2) \\ 8Dx_1^3y_1^3+6x_1y_1(x_1^2+Dy_1^2)^2 \end{pmatrix} \\
&= \begin{pmatrix} (x_1^2+x_1^2+4)^3+12x_1^2(x_1^2+4)(x_1^2+x_1^2+4) \\ 8x_1^3y_1(x_1^2+4)+6x_1y_1(x_1^2+x_1^2+4)^2 \end{pmatrix} \\
&= \begin{pmatrix} 32((x_1^2+1)^2(x_1^2+4)-2) \\ 32x_1y_1(x_1^2+1)(x_1^2+3) \end{pmatrix}
\end{aligned}$$

This completes the proof. □

References

[1] Ahmet Tekcan, The Pell Equation $x^2 - Dy^2 = \pm 4$, *Applied Mathematical Sciences*, **1**(8), 2007, 363-369

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