TL-Moments and L-Moments Estimation for the Generalized Pareto Distribution

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Abstract
In this paper, the trimmed L-moments (TL-moments) and L-moments of the Generalized Pareto distribution (GPD) up to arbitrary order will be derived and used to obtain the first four TL-moments and L-moments. TL-skewness, L-skewness, TL-kurtosis and L-kurtosis are handled for the GPD. Using the first two TL-moments and L-moments, the unknown parameters for the GPD can be estimated. A numerical illustrate for the new results will be given.

Keywords: GPD, TL-moments, L-moments, skewness, kurtosis, Method of TL-moments and L-moments estimation, Beta function, Gamma function, Order statistics

1 Introduction
The method of L-moment estimators have recently appeared. Hosking (1990) gives estimators for log-normal, gamma and generalized extreme value distributions. L-moment estimators for generalized Rayleigh distribution was introduced by Kundu and Raqab (2005). Karvanen (2006) applied the method of L-moment estimators to estimate the parameters of polynomial quantile mixture. He introduced the mixture composed of two parametric families, are the normal-polynomial quantile and Cauchy-polynomial quantile. The standard method to compute the L-moment estimators is to equate the sample L-moments with the corresponding population L-moments. A population L-moment $L_r$ is defined to be a certain linear function of the expectations of
the order statistics \(Y_{1:r}, Y_{2:r}, \ldots, Y_{r:r}\) in a conceptual random sample of size \(r\) from the underlying population. For example, \(L_1 = E(Y_{1:1})\), which is the same as the population mean, is defined in terms of a conceptual sample of size \(r = 1\), while \(L_2 = (1/2)E(Y_{2:2} - Y_{1:2})\), an alternative to the population standard deviation, is defined in terms of a conceptual sample of size \(r = 2\). Similarly, the L-moments \(L_3\) and \(L_4\) are alternatives to the un-scaled measures of skewness and kurtosis \(\mu_3\) and \(\mu_4\) respectively. See Silito (1969). Compared to the conventional moments, L-moments have lower sample variances and are more robust against outliers. Elamir and Seheult (2003) introduced an extension of L-moments called TL-moments. TL-moments are more robust than L-moments and exist even if the distribution does not have a mean, for example the TL-moments are existed for Cauchy distribution. Abdul-Moniem (2007) applied the method of L-moment and TL-moment estimators to estimate the parameters of exponential distribution. The following formula gives the \(r^{th}\) TL-moments (see Elamir and Seheult (2003)).

\[
L_r^{(t)} = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r - 1}{k} E(Y_{r+t-k:r+2t}),
\]

(1)

where \(r\) and \(t\) take the values 1, 2, 3, \ldots. Note that the \(r^{th}\) L-moments can be obtained by taking \(t = 0\). The GPD is defined by Abd Elfattah et. al (2007). They derived some well know distributions as a special cases from GPD. GPD has the following probability density function form:

\[
f(y; \alpha, \theta, \lambda, \delta) = \frac{\delta \alpha}{\theta} \left(\frac{y - \lambda}{\theta}\right)^{\delta-1} \left[1 + \left(\frac{y - \lambda}{\theta}\right)^\delta\right]^{-(\alpha+1)};
\]

\(y \geq \lambda > 0, \alpha, \theta & \delta > 0\)

(2)

where \(\theta\) is the scale parameter, \(\lambda\) is the location parameter and \((\alpha, \delta)\) are the shape parameters. The corresponding cumulative distribution function is

\[
F(y; \alpha, \theta, \lambda, \delta) = 1 - \left[1 + \left(\frac{y - \lambda}{\theta}\right)^\delta\right]^{-\alpha}.
\]

(3)

The main aim of this paper is to derive TL-moments and L-moments of the GPD up to arbitrary order and using it to estimate the unknown parameters. This paper is organized as follows: in Section 2, we introduced population TL-moments and TL-moment estimators for the GPD. The population L-moments and L-moment estimators for the GPD was presented in Section 3. In Section 4, A numerical illustrate for the new results will be given.

## 2 TL-moments for the GPD

In this section, the population TL-moment of order \(r\) for the GPD will be obtained. The sample TL-moments and the TL-moments estimators also discussed.
2.1 Population TL-moments

Using formula (1) and two functions (2) and (3), the TL-moment of order \( r \) for the GPD taking the following form

\[
L_r(t) = \frac{1}{r \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k}} \frac{(r+2t)!}{(r+t-k-1)!(t+k)!} I,
\]

where

\[
I = \int_0^\infty \frac{y^\delta \frac{\delta \alpha}{\theta} \left( \frac{y-\lambda}{\theta} \right)^{\delta-1} \left[ 1 + \frac{y-\lambda}{\theta} \right]^{-\alpha}}{\left[ 1 + \left( \frac{y-\lambda}{\theta} \right)^\delta \right]^{\alpha(t+k+1)}} dy
\]

By expanding \( [1 - [1 + (\frac{y-\lambda}{\theta})^\delta]^{-\alpha}]^{r+t-k-1} \) binomially, we get

\[
I = \sum_{j=0}^{r+t-k-1} \binom{r+t-k-1}{j} (-1)^j \int_0^\infty \frac{y^\delta \frac{\delta \alpha}{\theta} \left( \frac{y-\lambda}{\theta} \right)^{\delta-1} \left[ 1 + \frac{y-\lambda}{\theta} \right]^{-\alpha}}{\left[ 1 + \left( \frac{y-\lambda}{\theta} \right)^\delta \right]^{\alpha(t+k+j+1)}} dy
\]

let \( z = \left( \frac{y-\lambda}{\theta} \right)^\delta \), this led to \( y = \theta z + \lambda \) and \( |J| = \frac{\theta}{\delta (\frac{y-\lambda}{\theta})^{\delta-1}} \), then

\[
I = \sum_{j=0}^{r+t-k-1} \binom{r+t-k-1}{j} (-1)^j \int_0^\infty \frac{\theta z^\frac{\delta}{\theta} + \lambda}{\left[ 1 + z^{\frac{\delta}{\theta}} \right]^{\alpha(t+k+j+1)}} dy
\]

\[
= \sum_{j=0}^{r+t-k-1} \binom{r+t-k-1}{j} (-1)^j \alpha \left[ \theta \beta(1 + \frac{1}{\delta}, \alpha(t+k+j+1) - \frac{1}{\delta}) + \frac{\lambda}{\alpha(t+k+j+1)} \right]
\]

The \( L_r(t) \) becomes

\[
L_r(t) = \frac{1}{r \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k}} \frac{(r+2t)!}{(r+t-k-1)!(t+k)!} \sum_{j=0}^{r+t-k-1} \binom{r+t-k-1}{j} (-1)^j \alpha \left[ \theta \beta(1 + \frac{1}{\delta}, \alpha(t+k+j+1) - \frac{1}{\delta}) + \frac{\lambda}{\alpha(t+k+j+1)} \right] (4)
\]

where \( r, t = 1, 2, 3, \ldots \). Here, we take \( t = 1 \) (see Elamir and Seheult (2003)) then equation (4) becomes
\[ L_r^{(1)} = \frac{1}{r} \sum_{k=0}^{r-1} \left( -1 \right)^k \binom{r-1}{k} \frac{(r+2)!}{(r-k)!} \sum_{j=0}^{r-k} \binom{r-k}{j} \frac{(r-k)!}{(1+k)!} \]

\[ (-1)^{j} \alpha \left[ \theta \beta (1 + \frac{1}{\delta}, \alpha(k+j+2) - \frac{1}{\delta}) + \frac{\lambda}{\alpha(k+j+2)} \right] \] (5)

where \( r = 1, 2, 3, \ldots; \alpha, \lambda, \delta \) and \( \theta > 0 \). The first four TL-moments can be obtained by taking \( r = 1, 2, 3 \) and 4 in (5) as follows

\[ L_1^{(1)} = \frac{\theta \Gamma(1 + \frac{1}{\delta}) [3 \Gamma(3\alpha) \Gamma(2\alpha - \frac{1}{\delta}) - 2 \Gamma(2\alpha) \Gamma(3\alpha - \frac{1}{\delta})]}{\Gamma(2\alpha) \Gamma(3\alpha)} + \lambda \] (6)

\[ L_2^{(1)} = \frac{3\theta \Gamma(1 + \frac{1}{\delta}) [\Gamma(4\alpha) \Gamma(2\alpha - \frac{1}{\delta}) + \Gamma(2\alpha) \Gamma(4\alpha - \frac{1}{\delta})]}{\Gamma(2\alpha) \Gamma(4\alpha)} - \frac{6\theta \Gamma(1 + \frac{1}{\delta}) \Gamma(3\alpha - \frac{1}{\delta})}{\Gamma(3\alpha)} \] (7)

\[ L_3^{(1)} = \frac{10\theta \Gamma(1 + \frac{1}{\delta}) [\Gamma(3\alpha) \Gamma(2\alpha - \frac{1}{\delta}) - 4 \Gamma(2\alpha) \Gamma(3\alpha - \frac{1}{\delta})]}{3\Gamma(2\alpha) \Gamma(3\alpha)} + \frac{10\theta \Gamma(1 + \frac{1}{\delta}) [5 \Gamma(5\alpha) \Gamma(4\alpha - \frac{1}{\delta}) - 2 \Gamma(4\alpha) \Gamma(5\alpha - \frac{1}{\delta})]}{3\Gamma(4\alpha) \Gamma(5\alpha)} \] (8)

and

\[ L_4^{(1)} = \frac{15\theta \Gamma(1 + \frac{1}{\delta}) [\Gamma(4\alpha) \Gamma(2\alpha - \frac{1}{\delta}) + 15 \Gamma(2\alpha) \Gamma(4\alpha - \frac{1}{\delta})]}{4\Gamma(2\alpha) \Gamma(4\alpha)} + \frac{35\theta \Gamma(1 + \frac{1}{\delta}) [\Gamma(5\alpha) \Gamma(6\alpha - \frac{1}{\delta}) - 3 \Gamma(6\alpha) \Gamma(5\alpha - \frac{1}{\delta})]}{2\Gamma(5\alpha) \Gamma(6\alpha)} - \frac{25\theta \Gamma(1 + \frac{1}{\delta}) \Gamma(3\alpha - \frac{1}{\delta})}{\Gamma(3\alpha)} \] (9)

The TL-skewness \((\nabla_3)\) and TL-kurtosis \((\nabla_4)\) will be

\[ \nabla_3 = \frac{L_3^{(1)}}{L_2^{(1)}} = \frac{10 \Gamma(4\alpha) [\Gamma(3\alpha) \Gamma(2\alpha - \frac{1}{\delta}) - 4 \Gamma(2\alpha) \Gamma(3\alpha - \frac{1}{\delta})]}{9 \Psi(\alpha, \delta)} \]

\[ + \frac{10 \Gamma(2\alpha) \Gamma(3\alpha) [5 \Gamma(5\alpha) \Gamma(4\alpha - \frac{1}{\delta}) - 2 \Gamma(4\alpha) \Gamma(5\alpha - \frac{1}{\delta})]}{9 \Gamma(5\alpha) \Psi(\alpha, \delta)} \] (10)

and
\[ \nabla_4 = \frac{L_4^{(1)}}{L_2^{(1)}} = \frac{5\Gamma(4\alpha)\Gamma(3\alpha)\Gamma(2\alpha - \frac{1}{\delta}) - 20\Gamma(2\alpha)\Gamma(3\alpha - \frac{1}{\delta})}{12\Psi(\alpha, \delta)} \]
\[ + \frac{15\Gamma(2\alpha)\Gamma(3\alpha)\Gamma(4\alpha - \frac{1}{\delta}) - 14\Gamma(4\alpha)\Gamma(5\alpha - \frac{1}{\delta})}{12\Gamma(5\alpha)\Psi(\alpha, \delta)} \]
\[ + \frac{70\Gamma(2\alpha)\Gamma(3\alpha)\Gamma(4\alpha)\Gamma(6\alpha - \frac{1}{\delta})}{12\Gamma(6\alpha)\Psi(\alpha, \delta)} \]  
(11)

where
\[ \Psi(\alpha, \delta) = \Gamma(4\alpha)[\Gamma(3\alpha)\Gamma(2\alpha - \frac{1}{\delta}) - 2\Gamma(2\alpha)\Gamma(3\alpha - \frac{1}{\delta})] \]
\[ + \Gamma(2\alpha)\Gamma(3\alpha)\Gamma(4\alpha - \frac{1}{\delta}) \]

3 Sample TL-moments and TL-moment estimators

TL-moments can be estimated from a sample as linear combination of order statistics. Elamir and Seheult (2003) present the following estimator for sample TL-moments:

\[ l_r^{(t)} = \frac{1}{r} \sum_{i=t+1}^{n-t} \frac{\sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r+t-k-1} \binom{n-i}{t+k} x_{i:n}}{\binom{n}{r+2t}} \]  
(12)

where \( a \geq b \) for all \( \binom{a}{b} \) and \( x_{i:n} \) denotes the \( i^{th} \) order statistic in a sample of size \( n \). From (6), (7) and (12) with \( \alpha \) and \( \delta \) are known and \( t = 1 \), we can get the TL-moment estimator for \( \theta(\hat{\theta}_{TL}) \) and \( \lambda(\hat{\lambda}_{TL}) \) as follows

\[ l_1^{(1)} = \frac{6}{n(n-1)(n-2)} \sum_{i=2}^{n-1} (i-1)(n-i)x_{i:n} \]
\[ = \frac{\hat{\theta}_{TL}\Gamma(1 + \frac{1}{\delta})[3\Gamma(3\alpha)\Gamma(2\alpha - \frac{1}{\delta}) - 2\Gamma(2\alpha)\Gamma(3\alpha - \frac{1}{\delta})]}{\Gamma(2\alpha)\Gamma(3\alpha)} + \hat{\lambda}_{TL}, \]  
(13)
\( l_2^{(1)} = \frac{12}{n(n-1)(n-2)(n-3)} \left\{ \sum_{i=3}^{n-1} \binom{i-1}{2} \binom{n-i}{1} x_{i,n} - \sum_{i=2}^{n-2} \binom{i-1}{1} \binom{n-i}{2} x_{i,n}\right\} \\
 = 3\hat{\theta}_TL(1 + \frac{1}{\delta})[\Gamma(4\alpha)\Gamma(2\alpha - \frac{1}{\delta}) + \Gamma(2\alpha)\Gamma(4\alpha - \frac{1}{\delta})] \frac{\Gamma(2\alpha)\Gamma(4\alpha)}{\Gamma(2\alpha)\Gamma(4\alpha)} \\
 - \frac{6\hat{\theta}_TL(1 + \frac{1}{\delta})\Gamma(3\alpha - \frac{1}{\delta})}{\Gamma(3\alpha)} \tag{14} \\

By solving equations (13) and (14), we get \\
\( \hat{\theta}_TL = l_2^{(1)} = \frac{3\hat{\theta}_TL(1 + \frac{1}{\delta})[\Gamma(4\alpha)\Gamma(2\alpha - \frac{1}{\delta}) + \Gamma(2\alpha)\Gamma(4\alpha - \frac{1}{\delta})]}{\Gamma(2\alpha)\Gamma(4\alpha)} \frac{\Gamma(2\alpha)\Gamma(4\alpha)}{\Gamma(2\alpha)\Gamma(4\alpha)} \frac{6\hat{\theta}_TL(1 + \frac{1}{\delta})\Gamma(3\alpha - \frac{1}{\delta})}{\Gamma(3\alpha)} \), \tag{15} \\

and \\
\( \hat{\lambda}_TL = l_1^{(1)} - \frac{\hat{\theta}_TL(1 + \frac{1}{\delta})\Gamma(3\alpha)\Gamma(2\alpha - \frac{1}{\delta}) - 2\Gamma(2\alpha)\Gamma(3\alpha - \frac{1}{\delta})}{\Gamma(2\alpha)\Gamma(3\alpha)} \) \tag{16} \\

4 L-moments for the GPD

In this section, the population L-moment of order \( r \) for the GPD as a special case from formula (4) will be introduced. Sample L-moments and L-moments estimators also studied.

4.1 Population L-moments

Here, the population L-moment of order \( r \) for the GPD as a special case form (4) by taking \( t = 0 \) will be \\
\( L_r = \sum_{k=0}^{r-1} (-1)^k r! (r-1) \binom{r-1}{k} \sum_{j=0}^{r-k-1} \binom{r-k-1}{j} (-1)^j \)) \\
\( \alpha[\theta\beta(1 + \frac{1}{\delta}, \alpha(k + j + 1) - \frac{1}{\delta}) + \frac{\lambda}{\alpha(k + j + 1)}] \) \tag{17} \\

The first four L-moments can be obtained by taking \( r = 1, 2, 3 \) and 4 in (17) as follows \\
\( L_1 = \frac{\theta\Gamma(1 + \frac{1}{\delta})\Gamma(\alpha - \frac{1}{\delta})}{\Gamma(\alpha)} + \lambda \) \tag{18}
The L-skewness ($\tau_3$) and L-kurtosis ($\tau_4$) will be

$$\tau_3 = \frac{L_3}{L_2} = \frac{\Gamma(2\alpha)\Gamma(3\alpha)\Gamma(\alpha - \frac{1}{\delta}) - 3\Gamma(\alpha)\Gamma(3\alpha)\Gamma(2\alpha - \frac{1}{\delta})}{\Gamma(3\alpha)[\Gamma(2\alpha)\Gamma(\alpha - \frac{1}{\delta}) - \Gamma(\alpha)\Gamma(2\alpha - \frac{1}{\delta})]} + \frac{2\Gamma(\alpha)\Gamma(2\alpha)\Gamma(3\alpha - \frac{1}{\delta})}{\Gamma(3\alpha)[\Gamma(2\alpha)\Gamma(\alpha - \frac{1}{\delta}) - \Gamma(\alpha)\Gamma(2\alpha - \frac{1}{\delta})]}$$

(22)

and

$$\tau_4 = \frac{L_4}{L_2} = \frac{\Gamma(2\alpha)\Gamma(\alpha - \frac{1}{\delta}) - 6\Gamma(\alpha)\Gamma(2\alpha - \frac{1}{\delta})}{\Gamma(2\alpha)\Gamma(\alpha - \frac{1}{\delta}) - \Gamma(\alpha)\Gamma(2\alpha - \frac{1}{\delta})} + \frac{5\Gamma(\alpha)\Gamma(2\alpha)[2\Gamma(4\alpha)\Gamma(3\alpha - \frac{1}{\delta}) - \Gamma(3\alpha)\Gamma(4\alpha - \frac{1}{\delta})]}{\Gamma(3\alpha)\Gamma(4\alpha)\Gamma(2\alpha)\Gamma(\alpha - \frac{1}{\delta}) - \Gamma(\alpha)\Gamma(2\alpha - \frac{1}{\delta})}$$

(23)

4.2 Sample L-moments and L-moment estimators

Sample L-moments can be estimated from (12) by taking $t = 0$ as follows

$$l_r = \frac{1}{r} \sum_{i=1}^{n} \frac{\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}}{\binom{n}{r}} x_{i:n}$$

(24)

where $x_{i:n}$ as above. From (18), (19) and (24) with $\alpha$ and $\delta$ are known, the L-moment estimator for $\theta(\hat{\theta}_L)$ and $\lambda(\hat{\lambda}_L)$ will be

$$l_1 = \frac{1}{n} \sum_{i=1}^{n} x_{i:n} = \bar{x} = \frac{\hat{\theta}_L\Gamma(1+\frac{1}{\delta})\Gamma(\alpha - \frac{1}{\delta})}{\Gamma(\alpha)} + \hat{\lambda}_L,$$

(25)

and
\[ l_2 = \frac{2}{n(n-1)} \sum_{i=1}^{n} (i-1)x_{i,n} - \bar{x} \]

\[ \hat{\theta}_L \Gamma(1 + \frac{1}{\beta})[\Gamma(2\alpha)\Gamma(\alpha - \frac{1}{\beta}) - \Gamma(\alpha)\Gamma(2\alpha - \frac{1}{\beta})] \]

\[ \Gamma(\alpha) \Gamma(2\alpha) \] (26)

By solving equations (25) and (26), we get

\[ \hat{\theta}_L = \frac{l_2 \Gamma(\alpha)\Gamma(2\alpha)}{\Gamma(1 + \frac{1}{\beta})[\Gamma(2\alpha)\Gamma(\alpha - \frac{1}{\beta}) - \Gamma(\alpha)\Gamma(2\alpha - \frac{1}{\beta})]} \]

(27)

and

\[ \hat{\lambda}_L = l_1 - \frac{\hat{\theta}_L \Gamma(1 + \frac{1}{\beta})\Gamma(\alpha - \frac{1}{\beta})}{\Gamma(\alpha)} \]

(28)

### 5 A numerical illustration

By generating samples of size 10(10)40 with 10000 replications. Applying the program of Mathcad (2001), the estimates and their mean square error (MSE) of the unknown parameters \( \theta \) and \( \lambda \) using equations (15), (16), (27) and (28) are computed. Table (1) presents the estimates of \( \theta \) and \( \lambda \) and its MSEs using the exact value of \( \lambda = 2 \) with different values of \( \theta = 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4 \) and 1.6.

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<th>( \lambda )</th>
<th>( \hat{\lambda} )</th>
<th>( \hat{\theta} )</th>
<th>( \hat{\theta} )</th>
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<td>2.0004(0.0011)</td>
<td>1.9998(0.0008)</td>
<td>2.0003(0.0006)</td>
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<td>1.9998(0.0032)</td>
<td>2.0007(0.0022)</td>
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<tr>
<td>0.6</td>
<td>1.9989(0.0072)</td>
<td>1.9993(0.0028)</td>
<td>1.9998(0.0018)</td>
<td>2.0004(0.0013)</td>
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<td>2.0005(0.0115)</td>
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Table (1) Continued

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<td>1.6040(0.2408)</td>
<td>1.5940(0.1523)</td>
</tr>
</tbody>
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The MSEs are reported within brackets against each estimates.

From Table (1), one can show that

- The values of MSEs decrease as $n$ increases.

- The values of MSEs for $\hat{\lambda}_{TL}$ and $\hat{\theta}_{TL}$ are smaller than the corresponding values for $\hat{\lambda}_L$ and $\hat{\theta}_L$.

- The values of MSEs for $\hat{\lambda}_{TL}, \hat{\theta}_{TL}, \hat{\lambda}_L$ and $\hat{\theta}_L$ increase as the exact value of $\theta$ increases.

References


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