Bilateral Contact with Tresca’s Friction Law
with Internal State Variables

N. Lebri ¹, S. Djabi and S. Boutechebak

Department of Mathematics, Faculty of Sciences
University of Farhat Abbas, Setif, 19000, Algeria

Abstract

The subject of this work is the study of a value problem describing the quasistatic evolution of semilinear retetype viscoplastic models with internal state variables, and we suppose the problem of Tresca’s Friction Law at the presence of recal forces.

The existence and uniqueness of the solution is proved using results of evolutionary variational inequalities and a fixed point theorem.

Mathematics Subject Classification: 74M15, 74S05, 65M60

Keywords: viscoplasticity, Trasca’s law, variational inequality, fixed point

1 Introduction

For the bilateral contact problem studied in this paper, the friction is modelled by tesca’s law with internal state variables.

So, in section 2 contains the basic notations on the functional spaces and some hypothesis used in the following .

Section 3, mechanical problem a fixed is stated together with his variational formulation, in section 4 using results of evolutionary variational inequalities and a fixed point theorem , an existence and uniqueness result is obtained.

¹nem_mat2000@yahoo.fr
2 Statement of the problem. Hypothesis

We assume the contact is bilateral, i.e. there is no loss of the contact during the process. Thus, the normal displacement $u_\nu$ vanishes on $\Gamma_3$ at any time. We model the friction with the tresca friction law.

Let $\Omega \subset \mathbb{R}^N (N = 1, 2, 3)$ a bounded domain and whose boundary $\Gamma$ assumed to be sufficiently smooth, is partitionned into three disjoint measurable parts $\Gamma_1, \Gamma_2$ and $\Gamma_3$. Let meas $\Gamma_1 > 0$, let $T > 0$ be a time. Let $M$ be a natural number; we consider the following mixed problem:

\begin{align*}
(1) & \quad \dot{\sigma} = \xi(\varepsilon(\dot{u})) + G(\sigma, u, k) \quad \text{in } \Omega \times [0, T] \\
(2) & \quad \dot{k} = \phi(\varepsilon(u), \sigma, k) \quad \text{in } \Omega \times [0, T] \\
(3) & \quad \text{Div} \sigma + f_0 = 0 \quad \text{in } \Omega \times [0, T] \\
(4) & \quad u = 0 \quad \text{on } \Gamma_1 \times [0, T] \\
(5) & \quad \sigma_\nu + \Phi(x)u = f_2 \quad \text{on } \Gamma_2 \times [0, T] \\
(6) & \quad \begin{cases}
\nu = 0, & \|\sigma_r\| \leq g, \\
\|\sigma_r\| < g \Rightarrow \dot{u}_r = 0, \\
\|\sigma_r\| = g \Rightarrow \exists \lambda > 0 = -\lambda \dot{u}_r
\end{cases} \quad \text{on } \Gamma_3 \times [0, T] \\
(7) & \quad u(0) = u_0, \sigma(0) = \sigma_0, k(0) = k_0 \quad \text{in } \Omega
\end{align*}

in which the unknowns are the functions $u : \Omega \times (0, T) \to \mathbb{R}^N$, $\sigma : \Omega \times (0, T) \to S_N$ and $k : \Omega \times (0, T) \to \mathbb{R}^M$ in which $k$ may be interpreted as an internal state variable and $\xi$, $k$ and $G$ are given constitutive functions.

In (1)-(7), $u$ represents the displacement function, $\sigma$ represents the stress function, $\varepsilon(u)$ denotes the small strain tensor, $f_0$ is given body force, $f_2$ is given boundary data, and $u_0$, $\sigma_0$ and $k_0$ are the initial data. In (6), $g \geq 0$ is the friction bound function, i.e., magnitude of the limiting friction traction at which slip begins. The strict inequality in (6) holds in the stick zone and the equality holds in the slip zone. The equation (5) means that the Cauchy vector $\sigma_\nu$ is proportional on the displacement.

Viscoplastic models of the form (1),(2) are used in order to model the behaviour of real bodies for which the plastic rate of deformation depends also on an internal state variable.
Some of the internal state variables considered by many authors are the plastic strain, a number of tensor variables that takes into account the spatial display of dislocation for the internal state variables. Here we suppose that $k$ is a vector-valued function which satisfies (2) were $\phi$ is a given function.

In order to obtain variational formulations for the problem (1)-(7), let us consider the following notations:

$$H = \{ v = (v_i) \mid v_i \in L^2(\Omega), i = 1, N \}$$
$$H_1 = \{ v = (v_i) \mid v_i \in H^1(\Omega), i = 1, N \}$$
$$V = \{ v \in H_1 : v = 0 \text{ a.e on } \Gamma_1 \}$$
$$V_1 = \{ v \in V / v_\nu = 0 \text{ a.e on } \Gamma_3 \}$$
$$Q = \{ \tau = (\tau_{ij}) \mid (\tau_{ij}) \in L^2(\Omega), 1 \leq i, j \leq p \}$$
$$Q_1 = \{ \tau \in Q, \text{ Div } \tau \in H \}$$

The spaces $H, H_1, V, V_1, Q, Q_1$ are real Hilbert spaces endowed with the canonical inner products denoted by $\langle ., . \rangle_H, \langle ., . \rangle_{H_1}, \langle ., . \rangle_V, \langle ., . \rangle_{V_1}, \langle ., . \rangle_Q, \langle ., . \rangle_{Q_1}$ respectively.

We assume that the force and traction satisfy

$$f_0 \in W^{1,\infty}(0, T, H), f_2 \in W^{1,\infty}(0, T, H_{\Gamma_2}).$$

and the friction bound satisfies

$$g \in L^\infty(\Gamma_3), g \geq 0 \text{ on } \Gamma_3$$

Denote by $f(t)$ the element of $V_1$ giving by

$$\langle f(t), v \rangle_V = \int_\Omega f_0.v \, ds + \int_{\Gamma_2} f_2.v \, da \quad \forall v \in V_1$$

For all $t \in [0, T]$ let $j : V_1 \to \mathbb{R}_+$ be the functional

$$j(v) = \int_{\Gamma_2} g \| v_\tau \| \, da.$$  

For all $t \in [0, T],$ we note that conditions (8) and (9) imply

$$f \in W^{1,\infty}(0, T, V_1)$$

and

$$j(v) \leq c \| g \|_{L^\infty(\Gamma_3)} \| v \| \quad \forall v \in V_1$$

We also introduce the notation $\Sigma(t)$ defined by:
Finally, assume that the initial data satisfy
\[
\Sigma(0) = \{ \tau \in \mathbb{Q} : \langle \tau, \varepsilon(v) \rangle_{\mathbb{Q}} + \langle \Phi u_0, v \rangle_{L^2(\Gamma_2)} + j(v) \geq \langle f(t), v \rangle_V, \forall v \in V_1 \},
\]

We have the following result.

3 Variational formulation

Lemma 1 If \( u \) and \( \sigma \) are regular functions satisfying (3)-(6) then for all \( t \in [0, T] \),
\[
u(t) \in V_1, \quad \langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_{\mathbb{Q}} + \langle \Phi u(t), v - \dot{u}(t) \rangle_{L^2(\Gamma_2)} + j(v) \geq \langle f(t), v - \dot{u}(t) \rangle_V, \forall v \in V_1
\]

Proof: Let \( t \in [0, T] \) and \( v \in V_1 \) be an arbitrary test function. Multiplying the equation (3) by \( v - \dot{u}(t) \) and integrating by parts, we obtain:
\[
\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_{\mathbb{Q}} = \int_{\Omega} f_0(t)(v - \dot{u}(t))ds + \int_{\Gamma} \sigma \nu(v - \dot{u}(t))da
\]
Using the boundary condition (4) and (5) we can replace the right hand side of the above equality by
\[
\langle f(t), v - \dot{u}(t) \rangle_V - \int_{\Gamma_2} \Phi u(v - \dot{u}(t))da + \int_{\Gamma_3} \sigma \nu(v - \dot{u}(t))da
\]
From the contact boundary condition (6) we derive the relation
\[
\sigma \tau(t). \dot{u}_\tau(t) = -g \| \dot{u}_\tau(t) \| \quad \text{on} \Gamma_3
\]
Thus, on \( \Gamma_3 \),
\[
\sigma \nu(v - \dot{u}(t)) = \sigma \nu(t)(v_v - \dot{v}_v(t)) + \sigma \tau(t). (v_\tau - \dot{v}_\tau(t))
\]
\[
= \sigma \tau(t). v_\tau - \sigma \tau(t). \dot{v}_\tau(t)
\]
\[
= \sigma \tau(t). v_\tau + g \| \dot{v}_\tau(t) \|
\]
\[
\geq g(\| \dot{u}_\tau(t) \| - \| v_\tau \|).
\]
Therefore,
\[
\int_{\Gamma_3} \sigma \nu(v - \dot{u}(t))da \geq j(\dot{u}(t)) - j(v),
\]
and we have
\[ \langle \sigma(t), \varepsilon(v) - \varepsilon(\hat{u}(t)) \rangle_Q + \langle \Phi w, v - \hat{u}(t) \rangle_{L^2(\Gamma_2)^N} \geq \langle f(t), v - \hat{u}(t) \rangle_V + j(\hat{u}(t)) - j(v). \]
i.e the inequality in (12) holds.
The previous lemma leads us to consider the following weak formulations of the frictional problem (1) – (7).

**Problem 2** Find a displacement field \( u : [0, T] \to \mathbb{V}_1 \), a stress field \( \sigma : [0, T] \to \mathbb{Q}_1 \) and \( k : [0, T] \to \mathbb{R}^M \)

\[ \dot{\sigma}(t) = \xi(\varepsilon(\hat{u})) + G(\sigma, u, k) \quad \text{a.e } t \in (0, T) \]

\[ \dot{k} = \phi(\varepsilon(u), \sigma, k) \]

\[ \langle \sigma(t), \varepsilon(v) - \varepsilon(\hat{u}(t)) \rangle_Q + \langle \Phi u(t), v - \hat{u}(t) \rangle_{L^2(\Gamma_2)^N} + j(v) - j(\hat{u}(t)) \geq \langle f(t), v - \hat{u}(t) \rangle_V \quad \forall v \in \mathbb{V}_1 \]

\( u(0) = u_0, \ \sigma(0) = \sigma_0, k(0) = k_0 \)

### 4 Existence and Uniqueness results

In the study of the problem (1)–(7), we consider the following assumptions:

\[ \xi : \Omega \times S_N \to S_N \]

\[ (a) \ \xi_{ijkh} \in L^\infty(\Omega) \text{ for all } i, j, k, h = 0, \ldots, \overline{N} \]

\[ (b) \ \xi_{\sigma, \tau} = \xi_{\sigma, \tau} \forall \sigma, \tau \in S_N. \]

\[ (c) \ \exists \alpha > 0/\xi_{\sigma, \tau} \geq \alpha |\sigma|^2 \forall \sigma \in S_N. \]

\[ G : \Omega \times S_N \times S_N \times \mathbb{R}^M \to S_N \]

\[ (a) \ \exists \tilde{L} > 0 \text{ that } \forall \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in S_N, \ k_1, k_2 \in \mathbb{R}^M \text{ a.e in } \Omega \]

\[ |G(x, \sigma_1, \varepsilon_1, k_1) - G(x, \sigma_2, \varepsilon_2, k_2)| \leq \tilde{L}(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |k_1 - k_2|) \]

\[ (b) \ x \to G(x, \sigma, \varepsilon, k) \text{ is measurable function with respect to the Lebesgue measure on } \Omega \text{ for all } \sigma, \varepsilon \in S_N, \ k \in \mathbb{R}^M \]

\[ (c) \ x \to G(x, 0, 0, 0) \in L^2(\Omega)^{N \times N}. \]

\[ \phi : \Omega \times S_N \times S_N \times \mathbb{R}^M \to \mathbb{R}^M \text{ and} \]

\[ (a) \exists C > 0 \text{ that } \forall \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in S_N, \ k_1, k_2 \in \mathbb{R}^M \text{ a.e in } \Omega \]

\[ |\phi(x, \sigma_1, \varepsilon_1, k_1) - \phi(x, \sigma_2, \varepsilon_2, k_2)| \leq C(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |k_1 - k_2|) \]

\[ (b) \ x \to \phi(x, \sigma, \varepsilon, k) \text{ is measurable function with respect to the Lebesgue measure on } \Omega \text{ for all } \sigma, \varepsilon \in S_N, \ k \in \mathbb{R}^M \]

\[ (c) x \to \phi(x, 0, 0, 0) \in L^2(\Omega)^M. \]
Φ : Γ₂ → S_N such that :

\[ \Phi(x)u.u \geq 0 \quad \text{on } Γ₂, \]
\[ Φ_{km} ∈ L₂(Γ₂) \quad \text{for } k, m = 1, ..., N \]

(23) \( k_0 ∈ L²(Ω)^M \)

The main result of this section is the following:

**Theorem 1 (1)** Assume (8), (9), (14), (19) and (20)-(23), the problem (2) have a unique solution

(24) \( u ∈ W^{1,∞}(0, T, V₁), σ ∈ W^{1,∞}(0, T, Q₁), k ∈ W^{1,∞}(0, T, L²(Ω)^M). \)

**Proof:** We start by the existence part.
Let \( X \) be the product Hilbert space \( X = Q × L²(Ω)^M \) and let \( η = (η₁, η₂) \) in \( C₀(0, T, X) \).
We define the function \( Z_η = Z₁_η + Z₂_η \) in \( C₀(0, T, X) \) by

(25) \( Z_η(t) = \int_0^t η(s)ds + Z₀ \) where
(26) \( Z₀ = (σ₀ - ξε(u₀), k₀) \)

Consider the following auxiliary variational problem

**Problem 2** Find a displacement field \( u_η : [0, T] → V₁ \), a stress field \( σ_η : [0, T] → Q₁ \) and \( k ∈ W^{1,∞}(0, T, L²(Ω)^M) \)

(27) \( σ_η(t) = ξ(ε(u_η)) + Z₁_η(t) \) in \( Ω × (0, T) \)
(28) \( \langle σ_η(t), ε(v) - ε(\dot{u}_η(t)) \rangle_Q + \langle Φu(t), v - \dot{u}(t) \rangle_{L²(Γ₂)^N} \) + \( j(v) - j(\dot{u}_η(t)) ≥ \langle f(t), v - \dot{u}_η(t) \rangle_V \quad \forall v ∈ V₁ \)
(29) \( u_η(0) = u₀, σ_η(0) = σ₀, k(0) = k₀. \)

**Lemma 2.** The variational problem (2) has a unique solution \( u_η ∈ W^{1,∞}(0, T, V₁), σ_η ∈ W^{1,∞}(0, T, Q₁) \).

**Proof:** Define a bilinear form \( a : V₁ × V₁ → \mathbb{R} \) by
\[ a(u, v) = \langle ξ(ε(u)), ε(v) \rangle_Q + \langle Φu(t), v - \dot{u}(t) \rangle_{L²(Γ₂)^N} \quad \forall u, v ∈ V₁ \]
it follows from the assumptions (19) on \( ξ \) that \( a(u, v) \) is continuous, symmetric
and $V_1$-elliptic. Also by (9) we see $j$ is a continuous seminorm on $V_1$.

Let $f_\eta : [0, T] \to V_1$ by

$$\langle f_\eta, v \rangle_V = \langle f(t), v \rangle_V - \langle Z_\eta^1(t), \varepsilon(v) \rangle_Q \quad \forall v \in V_1.$$ 

Since $f \in W^{1,\infty}([0, T], V_1)$ and $Z_\eta^1 \in W^{1,\infty}([0, T], Q)$, we have $f_\eta \in W^{1,\infty}([0, T], V_1)$.

Moreover, since $\sigma_0 = \xi \varepsilon(u_0) + Z_\eta^0 = \xi \varepsilon(u_0) + Z_\eta^1(0)$ by (26) and (25), the assumption $\sigma_0 \in \Sigma(0)$ from (14) can be written as

$$\langle \xi \varepsilon(u_0), \varepsilon(v) \rangle_Q + \langle Z_\eta^1(0), \varepsilon(v) \rangle_Q + j(v) \geq \langle f(0), v \rangle_V \quad \forall v \in V_1$$

Applying a theorem in ([3] page 70) we see that there exists a unique solution $u_\eta \in W^{1,\infty}(0, T, V_1)$ to the problem

$$u_\eta(0) = u_0$$

Let $\sigma_\eta \in W^{1,\infty}(0, T, Q)$ be given by (26). Then $(u_\eta, \sigma_\eta)$ is a solution for (27)-(29). From inequality (28) we obtain $\text{Div} \sigma_\eta + f_\eta = 0$ a.e in $\Omega \times (0, T)$ . Therefore, $\sigma_\eta \in W^{1,\infty}(0, T, Q_1)$.

Let $k_\eta \in L^\infty(0, T, L^2(\Omega)^M)$ be the function defined by

$$k_\eta = Z_\eta^2$$

Finally, the uniqueness part of the lemma follows from the unique solvability of the variational inequality problem (30)-(31).

We consider next an operator $\Lambda : L^\infty(0, T, X) \to L^\infty(0, T, X)$ defined by

$$\Lambda \eta(t) = (G(\sigma_\eta(t), \varepsilon(u_\eta), k_\eta(t)), \phi(\sigma_\eta(t), \varepsilon(u_\eta), k_\eta(t)))$$

for all $t \in [0, T]$.

Where $(u_\eta, \sigma_\eta, k_\eta)$ is solution of problem (3).

**Lemma 3.** The operator $\Lambda$ has a unique fixed point $\eta_* \in L^\infty(0, T, X)$.

**Proof.** Let $\eta_1, \eta_2 \in L^\infty(0, T, X)$ and denote

$u_i = u_{\eta_i}, \sigma_i = \sigma_{\eta_i}, Z_i = Z_{\eta_i}, Z_i = Z_1^1 + Z_2^2$

and $\eta_i = (\eta_i^1, \eta_i^2)$ for all $i = 1, 2$. Rewrite (27) and (28) for $u_1$ and $u_2$ as

$$a(u_1(t), v - \hat{u}_1(t)) + \langle Z_1^1, \varepsilon(v) - \varepsilon(\hat{u}_1(t)) \rangle_Q + j(v) - j(\hat{u}_1(t)) \geq \langle f(t), v - \hat{u}_1(t) \rangle_V$$

$$a(u_2(t), v - \hat{u}_2(t)) + \langle Z_2^1, \varepsilon(v) - \varepsilon(\hat{u}_2(t)) \rangle_Q + j(v) - j(\hat{u}_2(t)) \geq \langle f(t), v - \hat{u}_2(t) \rangle_V$$
For all \( v \in V_1 \), a.e on \((0, T)\). we take \( v = \dot{u}_1 \) in the first inequality, \( v = \dot{u}_2 \) in the second inequality, and add the two inequalities to obtain

\[
a(u_2 - u_1, \dot{u}_2 - \dot{u}_1) \leq - \langle Z_1^1 - Z_2^1, \varepsilon(\dot{u}_1(t)) - \varepsilon(\dot{u}_2(t)) \rangle_Q \quad \text{a.e. on } (0, T).
\]

Let \( t \in [0, T] \). Integrate the inequality from 0 to \( t \):

\[
\frac{1}{2} \frac{d}{dt} a(u_2 - u_1, u_2 - u_1) \leq - \langle Z_1^1 - Z_2^1, \varepsilon(u_1(t)) - \varepsilon(u_2(t)) \rangle_Q + \int_0^t \langle \eta_1^1(s) - \eta_2^1(s), \varepsilon(u_1(s)) - \varepsilon(u_2(s)) \rangle_Q ds
\]

Then,

\[
c \| u_1(t) - u_2(t) \|_V^2 \leq \| Z_1^1(t) - Z_2^1(t) \|_Q \| u_1(t) - u_2(t) \|_V + \int_0^t \| \eta_1^1(s) - \eta_2^1(s) \|_Q \| u_1(s) - u_2(s) \|_V ds.
\]

So

\[
\| Z_1^1(t) - Z_2^1(t) \|_Q \leq \int_0^t \| \eta_1^1(s) - \eta_2^1(s) \|_Q ds.
\]

Therefore, we have

\[
\| u_1(t) - u_2(t) \|_V^2 \leq c \int_0^t \| \eta_1^1(s) - \eta_2^1(s) \|_Q^2 ds + \int_0^t \| u_1(s) - u_2(s) \|_V^2 ds.
\]

Applying the Gronwall inequality, we obtain

\[
(34) \quad \| u_1(t) - u_2(t) \|_V^2 \leq c \int_0^t \| \eta_1^1(s) - \eta_2^1(s) \|_Q^2 ds
\]

By the definition (27) and (33) we have

\[
(35) \quad \| \sigma_1(t) - \sigma_2(t) \|_V^2 \leq c \int_0^t \| \eta_1^1(s) - \eta_2^1(s) \|_Q^2 ds
\]

Finally,

\[
\Lambda \eta_1(t) - \Lambda \eta_2(t) = (G(\sigma_1(t), \varepsilon(u_{\eta_1}), K_{\eta_1}(t)), \phi(\sigma_1(t), \varepsilon(u_{\eta_1}), K_{\eta_1}(t)) - (G(\sigma_2(t), \varepsilon(u_{\eta_2}), K_{\eta_2}(t)), \phi(\sigma_2(t), \varepsilon(u_{\eta_2}), K_{\eta_2}(t))
\]

Using now (20), (21), (34) and (33) we get
Lemma (3) follows from (36) and an application of Banach’s fixed point theorem.

Now we are ready to prove Theorem (1).

**Proof of theorem (1)**

**Existence.**

Let $\eta^* \in L^\infty(0, T, X)$ be the fixed point of $\Lambda$ and let $(u_{\eta^*}, \sigma_{\eta^*}, K_{\eta^*})$ be the solution of problem (3) for $\eta = \eta^*$. Using (27) and (25) we have

$$\dot{\sigma}_{\eta^*}(t) = \xi(\varepsilon(\dot{u}_{\eta^*}(t))) + \eta^*(t) \ a.e. \ t \in (0, T).$$

Since

$$\eta^*(t) = \Lambda \eta^*(t) = (G(\sigma_{\eta^*}(t), \varepsilon(u_{\eta^*}), K_{\eta^*}(t)), \phi(\sigma_{\eta^*}(t), \varepsilon(u_{\eta^*}), K_{\eta^*}(t))) \ a.e. \ t \in (0, T),$$

it follows that $(u_{\eta^*}, \sigma_{\eta^*}, K_{\eta^*})$ satisfies (16). Using now (28) and (29) we conclude that is a solution of problem (2) with regularity (24).

**Uniqueness.** The uniqueness part follows from the uniqueness of the fixed point of the operator $\Lambda$.

**References**


Received: September 25, 2007