An Inequality for the Psi Functions

Lingli Wu
School of Educational Science and Technology
Huzhou Teachers College, Huzhou 313000, P. R. China

Yuming Chu
Department of Mathematics
Huzhou Teachers College, Huzhou 313000, P. R. China
chuyuming@hutc.zj.cn

Abstract

For $x > 0$, let $\Gamma(x)$ be the Euler’s gamma function, and $\psi(x) = \Gamma'(x)/\Gamma(x)$ the psi function. In this paper, we prove that $|\psi^{(n)}(b) - \psi^{(n)}(a)| < (b - L(a, b))|\psi^{(n+1)}(b)| + (L(a, b) - a)|\psi^{(n+1)}(a)|$ for all $b > a > 0$ and $n = 0, 1, 2, \ldots$, where $L(a, b) = (b - a)/(\log b - \log a)$.

Mathematics Subject Classification: 33B15, 26D15

Keywords: Gamma function, psi function, GA-convex function, GA-concave function

1. Introduction

For real and positive values of $x$, the Euler’s gamma function and its logarithmic derivative $\psi$, the so-called psi function, are defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \quad (1.1)$$

For extensions of these functions to complex variables and for basic properties see [19].

Over the past half century many authors have established inequalities for these important functions (see [1-5,7-9,11,13,14,16,18] and the references therein). It was shown in [10,12] that gamma and psi functions inequalities have interesting applications in the theory of 0-1 matrices and in graph theory.

---

1The research is partly supported by the NSF of China under Grant No. 10471039.
2Corresponding author
For $b > a > 0$, the generalized logarithmic mean $L_p(a, b)$ of $a$ and $b$ is defined as
\[
L_p(a, b) = \begin{cases} 
\left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{1/p}, & p \neq -1, 0, \\
\log \frac{b}{a}, & p = -1, \\
\frac{1}{e} \left(\frac{b^p}{a^p}\right)^{1/(b-a)}, & p = 0.
\end{cases}
\]

It is well-known that $L_p(a, b)$ is a increasing function on $p$ for fixed $a$ and $b$. Denote $A(a, b) = L_1(a, b) = (b + a)/2$, $I(a, b) = L_0(a, b) = \left(\frac{b^p}{a^p}\right)^{1/(b-a)}/e$, $L(a, b) = L_{-1}(a, b) = (b - a)/(\log b - \log a)$, $G(a, b) = L_{-2}(a, b) = \sqrt{ab}$ be the arithmetic mean, identric mean, logarithmic mean and geometric mean of $a$ and $b$, respectively.

Recently, N. Batir [6] proved
\[
|\psi^{(n)}(b) - \psi^{(n)}(a)| < (b - a)|\psi^{(n+1)}(L_{-(n+2)}(a, b))|, \quad n = 1, 2, 3, \cdots. \tag{1.2}
\]

The purpose of this paper is to establish the following new upper bound for $|\psi^{(n)}(b) - \psi^{(n)}(a)|$:

**Theorem.** If $b > a > 0$, $n = 0, 1, 2, \cdots$, then
\[
|\psi^{(n)}(b) - \psi^{(n)}(a)| < (b - L(a, b))|\psi^{(n+1)}(b)| + (L(a, b) - a)|\psi^{(n+1)}(a)|.
\]

## 2. Lemmas and Proof of Theorem

First we introduce three definitions:

**Definition 2.1.** Let $I \subseteq R$ be an interval, $f : I \to R$ is called a convex(concave) function if $f(\alpha x + (1 - \alpha)y) \leq (\geq)\alpha f(x) + (1 - \alpha)f(y)$ for all $x, y \in I$ and $\alpha \in [0, 1]$.

**Definition 2.2.** Let $I \subseteq R$ be an interval, $f : I \to (0, +\infty)$ is called a logarithmically convex(concave) function if $f(\alpha x + (1 - \alpha)y) \leq (\geq)f(x)^{\alpha}f(y)^{1-\alpha}$ for all $x, y \in I$ and $\alpha \in [0, 1]$.

**Definition 2.3.** Let $I \subseteq (0, +\infty)$ be an interval, $f : I \to R$ is called a GA-convex(concave) function if $f(x^{\alpha}y^{1-\alpha}) \leq (\geq)\alpha f(x) + (1 - \alpha)f(y)$ for all $x, y \in I$ and $\alpha \in [0, 1]$.

Next we shall establish and introduce the following five lemmas, they are the key of the proof of our main result in this section.

**Lemma 2.1.** Let $b > a > 0$, $c \in (a, b)$. If $f : [a, b] \to R$ is a differentiable GA-convex(concave) function, then
\[
(b - a)f(c) + cf'(c)((\log b - \log c)(b - L(c, b)) - (\log c - \log a)(L(a, c) - a)) \\
\leq (\geq) \int_a^b f(x)dx \leq (\geq)(b - L(a, b))f(b) + (L(a, b) - a)f(a), \tag{2.1}
\]
with equality (for left or right hand side) if and only if \( f(x) = p \log x + q \) for some \( p, q \in \mathbb{R} \).

**Proof.** To prove the left hand side inequality in (2.1). For \( x \in [c, b] \), let \( c_1 = \log c \) and \( x_1 = \log x \). Taking \( g(t) = f(e^t) \), then \( g : [\log a, \log b] \rightarrow \mathbb{R} \) is a convex (concave) function because of \( f \) is a \( GA \)-convex (concave) function.

This and the Lagrange mean value theorem yield

\[
\frac{g(x_1) - g(c_1)}{x_1 - c_1} \geq (\leq) g'(c_1),
\]

(2.2)

this leads to

\[
f(x) - f(c) \geq (\leq) c f'(c)(\log x - \log c).
\]

(2.3)

Next let \( h(x) = \int_c^x f(t)dt - (x-c)f(c) - cf'(c)(x(\log x - \log c) - x + c), x \in [c, b] \).

Then \( h(c) = 0 \) and \( h'(x) \geq (\leq) 0 \) for \( x \in [c, b] \) by (2.3), this implies \( h(x) \geq (\leq) 0 \) for all \( x \in [c, b] \). Hence \( h(b) \geq (\leq) 0 \), this yields

\[
\int_c^b f(x)dx \geq (\leq) (b - c)f(c) + cf'(c)(\log b - \log c) - b + c
\]

\[
= (b - c)f(c) + cf'(c)(\log b - \log c)(b - L(c, b)).
\]

(2.4)

The similar argument as above gives

\[
\int_a^c f(x)dx \geq (\leq)(c - a)f(c) + cf'(c)(\log c - \log a)(a - L(a, c)).
\]

(2.5)

Combining (2.4) and (2.5) we can get the left hand side inequality in (2.1).

To prove the right hand side inequality in (2.1). For any \( x \in [a, b] \), taking \( y = (\log x - \log a)/(\log b - \log a) \). Then \( 0 \leq y \leq 1 \) and \( x = a^{1-y} b^y \), by the definition of \( GA \)-convex (concave) function and the transformation to variable of integration, we have

\[
\int_a^b f(x)dx = \int_0^1 f(a^{1-y} b^y)d(a^{1-y} b^y)
\]

\[
\leq (\geq) a \int_0^1 ((1 - y)f(a) + y f(b))d(\frac{b}{a})^y
\]

\[
= b f(b) - a f(a) - L(a, b)(f(b) - f(a))
\]

\[
= (b - L(a, b))f(b) + (L(a, b) - a) f(a)
\]

(2.6)

At last, from the above argument, it is easy to see that the left or right hand side inequality becomes equality if and only if \( f(e^x) = px + q \) for some \( p, q \in \mathbb{R} \), namely, \( f(x) = p \log x + q \).

**Lemma 2.2.** (see[17]) Let \( I \subseteq (0, +\infty) \) be an interval. If \( f : I \rightarrow \mathbb{R} \) is a twice differentiable function, then \( f \) is a \( GA \)-convex (concave) function in \( I \) if and only if \( xf'(x) + x^2 f''(x) \geq (\leq) 0 \) for all \( x \in I \).
Lemma 2.3. (see [11,20]) For any \( x > 0 \), the following statements are true:

(a) \( 2\psi'(x) + x\psi''(x) < \frac{1}{x} \), \hspace{1cm} (2.7)

(b) \( \psi'(x) > \frac{1}{x} + \frac{1}{2x^2} \). \hspace{1cm} (2.8)

Lemma 2.4. (see [15]) Let \( x > 0 \), \( n = 0, 1, 2, \cdots \). If \( 0 \leq \alpha \leq n \), then

\[ x|\psi^{(n+1)}(x)| - \alpha|\psi^{(n)}(x)| > 0. \hspace{1cm} (2.9) \]

Lemma 2.5. If \( b > a > 0 \), \( n = 0, 1, 2, \cdots \), then \((-1)^n\psi^{(n)}(x)\) is a GA-concave function in \([a, b]\).

Proof. If \( n = 0 \), then (2.7) and (2.8) lead to

\[
\begin{align*}
2\psi'(x) + x\psi''(x) &< \frac{1}{x} \\
< x\left(\frac{1}{x} - \psi'(x)\right) \\
< -\frac{1}{2x} < 0.
\end{align*}
\]

This and Lemma 2.2 imply that \( \psi(x) \) is a GA-concave function.

Next we assume that \( n \geq 1 \). It is well-known that \( \log \Gamma(x) = -cx + \sum_{k=1}^{\infty} \left(\frac{x}{k} - \log(1 + \frac{x}{k})\right) - \log x \), where \( c = 0.577215 \cdots \) is the Euler’s constant. From this we can get

\[
\psi^{(n)}(x) = (-1)^{n+1}n! \sum_{k=0}^{\infty} \frac{1}{(k + x)^{n+1}}. \hspace{1cm} (2.10)
\]

(2.10) and Lemma 2.4 lead to

\[
\begin{align*}
x((-1)^n\psi^{(n)}(x))' + x^2((-1)^n\psi^{(n)}(x))'' &\hspace{1cm} (n + 1)!
\sum_{k=0}^{\infty} \frac{1}{(k + x)^{n+2}} - (n + 2)!x^2 \sum_{k=0}^{\infty} \frac{1}{(k + x)^{n+3}} \\
&= -x \left(x|\psi^{(n+2)}(x)| - |\psi^{(n+1)}(x)|\right) \\
&< 0. \hspace{1cm} (2.11)
\end{align*}
\]

(2.11) and Lemma 2 imply that \((-1)^n\psi^{(n)}(x)\) is a GA-concave function in \([a, b]\).

Now we can prove our Theorem.

Proof of Theorem. By Lemma 2.5 we know that \((-1)^n\psi^{(n+1)}(x)\) is a GA-convex function in \([a, b]\), making use of Lemma 2.1 and (2.10) we get

\[
\left|\psi^{(n)}(b) - \psi^{(n)}(a)\right|
\]
An inequality for the Psi functions

\[ \left| \int_a^b \psi^{(n+1)}(x) \, dx \right| \]
\[ = \int_a^b (-1)^n \psi^{(n+1)}(x) \, dx \]
\[ < (b - L(a, b))(-1)^n \psi^{(n+1)}(b) + (L(a, b) - a)(-1)^n \psi^{(n+1)}(a) \]
\[ = (b - L(a, b)) \left| \psi^{(n+1)}(b) \right| + (L(a, b) - a) \left| \psi^{(n+1)}(a) \right|. \]

References


Received: August 16, 2007