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On the Precise Asymptotics in Complete Moment Convergence of Moving Average Processes under NA Random Variables[1]

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Abstract

Let $\{\varepsilon_i | -\infty < i < \infty\}$ be a sequence of identically distributed negatively associated random variables and $\{a_i | -\infty < i < \infty\}$ a sequence of real numbers with $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. Set the moving average $S_n = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n}\sum_{i=-\infty}^{\infty} a_{i+k}\varepsilon_i,\ k \geq 1$, precise asymptotics $\sum_{n=1}^{\infty} n^{\frac{r}{p}-2} E(|S_n| - \varepsilon n^{\frac{1}{p}})_+ = \frac{p}{r-p} E(|N|)^{\frac{2(r-p)}{2-p}}$ as $\varepsilon \searrow 0$ are established, where N has a normal distribution with mean 0 and variance $\tau^2 = \sigma^2(\sum_{i=-\infty}^{\infty} a_i)^2$.

Mathematics Subject Classification: 60F15

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1. Introduction and main results

Let $\{\varepsilon_i | -\infty < i < \infty\}$ be a sequence of identically distributed random variables and $\{\varepsilon_i | -\infty < i < \infty\}$ a sequence of real numbers with $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, put

$$X_k = \sum_{i=-\infty}^{\infty} a_{i+k}\ \varepsilon_i,\ k \geq 1. \tag{1.1}$$

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When $\{\varepsilon_i| -\infty < i < \infty\}$ is a sequence of some suitable conditions there have been some authors who studied limit properties for the moving average processes $\{X_k|k \geq 1\}$. In particular, Ibragimov(1962) had established the central limit theorem for $\{X_k|k \geq 1\}$. Burton and Dehling (1990) had obtained large deviation principle, Yang(1996) had obtained the central limit theorem and the law of the iterated logarithm, Li et al.(1992) and Zhang(1996) had obtained the result on the complete convergence and Hall(1992) had obtained convergence rates in the central limit theorem for means of autoregressive and moving average sequences, etc. Moreover, many authors obtained a convergence of weighted sums of negatively associated random variables, including moving average processes, Yu and Wang (2002) and Baek, et al.(2003) obtained some general results on the convergence of moving average processes under dependent conditions. When $\{X_k|k \geq 1\}$ is a sequence of $i.i.d$ random variables with common distribution function F, mean 0 and positive, finite variance, Chen(1978) and Gut and Spătaru(2000) and Lanzinger and Stadtmüller(2004) obtained the precise asymptotics in the Baum-Katz law of large numbers as $\varepsilon \searrow 0$, Cheng et al.(2004) obtained precise asymptotics of partial sums, and Li, Li et al.(2004,2006) and Li, Nguyen and Rosalsky(2005)obtained a supplement to precise asymptotics in the law of the iterated logarithm and Chow(1988) obtained the complete moment convergence. Some of their results are as follows.

Theorem A. Suppose that $\{X_k|k \geq 1\}$ is a sequence of $i.i.d$ random variables with $EX_1 = 0$ and $0 < E{X_1}^2 = \gamma^2 < \infty$. Then, for $1 \leq p < r$,

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P(|\sum_{k=1}^{n} X_k| \geq \varepsilon n^{\frac{1}{p}}) = \frac{p}{r-p} E|N|^{\frac{2(r-p)}{2-p}},$$

where N has a normal distribution with mean 0 and variance γ^2.

Theorem B. Suppose that $\{X_k|k \geq 1\}$ is a sequence of $i.i.d$ random variables with $EX_1 = 0$. If $E(|X_1|^r + |X_1| \log(1 + |X_1|)) < \infty$, then for any $\varepsilon > 0$, $1 \leq p < 2$ and $r > p$,

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} E(|\sum_{k=1}^{n} X_k| - \varepsilon n^{\frac{1}{p}})_+ < \infty.$$

Our purpose in this paper is to show that this kind of result also holds for moving average processes under negatively associated random variables. First we shall give the definition of negatively associated random variables.

Definition. A finite family of random variables $\{\varepsilon_i | 1 \leq i \leq n\}$ is said to be negatively associated(NA) random variables if for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$, we have

$$Cov(f(\varepsilon_i,\ i \in A),\ g(\varepsilon_j,\ j \in B)) \leq 0,$$

whenever f and g are coordinate wise non-decreasing and the covariance exists. An infinite family is NA if every finite subfamily is NA.

Set $S_n = \sum_{k=1}^{n} X_k,\ n \geq 1$, where $\{X_k\}$ is as in (1.1). Our result is as follows.

Theorem 1.1. Suppose $\{X_k | k \geq 1\}$ is defined as (1.1), where$\{a_i | -\infty < i < \infty\}$ is a sequence of real numbers with $\sum_{i=-\infty}^{\infty} |a_i| < \infty$ and $\{\varepsilon_i | -\infty < i < \infty\}$ is a sequence of identically distributed NA random variables with $E\varepsilon_1 = 0,\ E{\varepsilon_1}^2 < \infty$. Suppose $E{\varepsilon_1}^3 < \infty$ and if $1 \leq p < r < 2$,

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n \geq 1} n^{\frac{r}{p}-2} E(|S_n| - \varepsilon n^{\frac{1}{p}})_+ = \frac{p}{r-p} E(|N|)^{\frac{2(r-p)}{2-p}}.$$

2. Some Lemmas

Lemma 1(Burton and Dehling(1990)). Let $\sum_{i=-\infty}^{\infty} a_i$ be an absolutely convergent series of real numbers with $a = \sum_{i=-\infty}^{\infty} a_i$ and $b = \sum_{i=-\infty}^{\infty} |a_i|$. Suppose $\phi : [-b,\ b] \longrightarrow R$ is a function satisfying the following conditions:
i) ϕ is bounded and continuous at a.
ii) There exist $\delta > 0$ and $C > 0$ such that for all $|x| \leq \delta,\ |\phi(x)| \leq C|x|$. Then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} \phi(\sum_{j=i+1}^{i+n} a_j) = \phi(a).$$

Remark. Taking $\phi(x) = |x|^q,\ q \geq 1$, from Lemma 1, we have

$$\lim_{n \to \infty} \sum_{i=-\infty}^{\infty} |\sum_{j=i+1}^{i+n} a_j|^q = |a|^q.$$

Lemma 2(Shao(2000)). Let $\{\varepsilon_i | 1 \leq i \leq n\}$ be a sequence of NA random variables with $E\varepsilon_i = 0$ and $E|\varepsilon_i|^p < \infty$ for some $1 \le p \le 2$. There exists constant $C_p > 0$ such that

$$E \max_{1\le k\le n} |\sum_{i=1}^{k} \varepsilon_i|^p \le C_p \sum_{i=1}^{k} E|\varepsilon_i|^p.$$

Lemma 3(Li(2006)). Let $\{\varepsilon_i| -\infty < i < \infty\}$ be a sequence of random variables with $E\varepsilon_1 = 0$ and $E{\varepsilon_1}^2 = \sigma^2$, and let $X_k = \sum_{i=-\infty}^{\infty} a_{i+k}\ \varepsilon_i,\ k \ge 1$, where $\{a_i| -\infty < i < \infty\}$ is a sequence of real numbers with $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. Then $\frac{S_n}{\tau\sqrt{n}} \xrightarrow{p} N(0,1)$, where $S_n = \sum_{k=1}^{n} X_k,\ \tau = \sigma \sum_{i=-\infty}^{\infty} a_i$.

Proof of Theorem 1.1

Let $a(\varepsilon) = \frac{-2p}{\varepsilon^{2-p}}$. Theorem 1.1 will be proved via three propositions.

Proposition 3.1. For any $1 \le p < r < 2$, we have

$$\lim_{\varepsilon\searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n\ge 1} n^{\frac{r}{p}-2} E(|N| - \varepsilon n^{\frac{1}{p}-\frac{1}{2}}) = \frac{p}{r-p} E|N|^{\frac{2(r-p)}{2-p}}$$

Proof. Let $y = \varepsilon x^{\frac{1}{p}-\frac{1}{2}}$ .Then

$$\begin{aligned}
& \lim_{\varepsilon\searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n\ge 1} n^{\frac{r}{p}-2} E(|N| - \varepsilon n^{\frac{1}{p}-\frac{1}{2}}) \\
= & \lim_{\varepsilon\searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n\ge 1} n^{\frac{r}{p}-2} \int_{\varepsilon n^{\frac{1}{p}-\frac{1}{2}}}^{\infty} P(|N| \ge t)dt \\
= & \lim_{\varepsilon\searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \int_{1}^{\infty} x^{\frac{r}{p}-2} \int_{\varepsilon x^{\frac{1}{p}-\frac{1}{2}}}^{\infty} P(|N| \ge t)dt\ dx \\
= & \frac{2p}{2-p} \lim_{\varepsilon\searrow 0} \int_{\varepsilon}^{\infty} y^{\frac{2(r-p)}{2-p}-2} \int_{y}^{\infty} P(|N| \ge t)dt\ dy \quad \text{by letting} \quad y = \varepsilon x^{\frac{1}{p}-\frac{1}{2}} \\
= & \frac{2p}{2-p} \lim_{\varepsilon\searrow 0} \int_{\varepsilon}^{\infty} P(|N| \ge t) \int_{\varepsilon}^{t} y^{\frac{2(r-p)}{2-p}-1} dy\ dt \\
= & \frac{2p}{2-p}\ \frac{2-p}{2(r-p)} \lim_{\varepsilon\searrow 0} \int_{\varepsilon}^{\infty} t^{\frac{2(r-p)}{2-p}} P(|N| \ge t)dt \\
= & \frac{p}{r-p}\ E(|N|)^{\frac{2(r-p)}{2-p}}
\end{aligned}$$

Proposition 3.2. For any $1 \le p < r < 2$ and $M > 1$, we have

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n \le a(\varepsilon)M} n^{\frac{r}{p}-2} |(E|S_n| - \varepsilon n^{\frac{1}{p}}) - E(|N| - \varepsilon n^{\frac{1}{p}-\frac{1}{2}})| = 0.$$

Proof. $\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n \le a(\varepsilon)M} n^{\frac{r}{p}-2} |E(|S_n| - \varepsilon n^{\frac{1}{p}}) - E(|N| - \varepsilon n^{\frac{1}{p}-\frac{1}{2}})|$

$$\begin{aligned}
&= \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n \le a(\varepsilon)M} n^{\frac{r}{p}-2} \Big| \int_0^\infty P(|S_n| \ge \varepsilon n^{\frac{1}{p}} + x)dx \\
&\quad - \int_0^\infty P(|N| \ge \varepsilon n^{\frac{1}{p}-\frac{1}{2}} + x)dx \Big| \\
&\le \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n \le a(\varepsilon)M} n^{\frac{r}{p}-2} \Big| \int_0^{n^{\frac{1}{p}}} P(|S_n| \ge \varepsilon n^{\frac{1}{p}} + x)dx \\
&\quad + \int_{n^{\frac{1}{p}}}^\infty P(|S_n| \ge \varepsilon n^{\frac{1}{p}} + x)dx \Big| - \Big| \int_0^{n^{\frac{1}{p}-\frac{1}{2}}} P(|N| \ge \varepsilon n^{\frac{1}{p}-\frac{1}{2}} + x)dx \\
&\qquad + \int_{n^{\frac{1}{p}-\frac{1}{2}}}^\infty P(|N| \ge \varepsilon n^{\frac{1}{p}-\frac{1}{2}} + x)dx \Big| \\
&\le \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n \le a(\varepsilon)M} n^{\frac{r}{p}-2} \Big| \int_0^{n^{\frac{1}{p}-\frac{1}{2}}} \Big| P(\frac{|S_n|}{\sqrt{n}} \ge \varepsilon n^{\frac{1}{p}-\frac{1}{2}} + x) \\
&\quad - P(|N| \ge \varepsilon n^{\frac{1}{p}-\frac{1}{2}} + x) \Big| dx + \int_{n^{\frac{1}{p}-\frac{1}{2}}}^\infty P(\frac{|S_n|}{\sqrt{n}} \ge \varepsilon n^{\frac{1}{p}-\frac{1}{2}} + x) \\
&\qquad - P(|N| \ge \varepsilon n^{\frac{1}{p}-\frac{1}{2}} + x) \Big| dx. \\
&=: \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n \le a(\varepsilon)M} n^{\frac{r}{p}-2} (I_1 + I_2).
\end{aligned}$$

Note that

$$\Delta_n = \sup_x |P(|S_n| > x n^{\frac{1}{2}}) - P(|N| > x),$$

then $\Delta_n \to 0$ as $n \to \infty$, from a well-known corollary of CLT.
For $p \ge 1$, we estimate I_1.

$$\begin{aligned}
I_1 &= \int_0^{n^{\frac{1}{p}-\frac{1}{2}}} |P(\frac{|S_n|}{\sqrt{n}} \ge \varepsilon n^{\frac{1}{p}-\frac{1}{2}} + x) - P(|N| \ge \varepsilon n^{\frac{1}{p}-\frac{1}{2}} + x)|dx \\
&\le n^{\frac{1}{p}-\frac{1}{2}} \sup_{0<x<\infty} |P(\frac{|S_n|}{\sqrt{n}} \ge x) - P(|N| \ge x)| \\
&\le n^{\frac{1}{p}-\frac{1}{2}} \Delta_n \longrightarrow 0 \;\; as \;\; n \to \infty.
\end{aligned}$$

So we get

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\varepsilon^{\frac{2(r-p)}{2-p}}\sum_{n\le a(\varepsilon)M} n^{\frac{r}{p}-2}I_1\\
\le\ &\lim_{\varepsilon\searrow 0}\varepsilon^{\frac{2(r-p)}{2-p}}\sum_{n\le [a(\varepsilon)M]} n^{\frac{r}{p}-2}I_1\\
=\ &\lim_{\varepsilon\searrow 0}\varepsilon^{\frac{2(r-p)}{2-p}}[a(\varepsilon)M]^{\frac{r}{p}-1}[a(\varepsilon)M]^{1-\frac{r}{p}}\sum_{n\le [a(\varepsilon)M]} n^{\frac{r}{p}-2}I_1\\
=\ &\lim_{\varepsilon\searrow 0}\varepsilon^{\frac{2(r-p)}{2-p}}[\varepsilon^{\frac{-2p}{2-p}}M]^{\frac{r}{p}-1}[a(\varepsilon)M]^{1-\frac{r}{p}}\sum_{n\le [a(\varepsilon)M]} n^{\frac{r}{p}-2}I_1\\
\le\ &\lim_{\varepsilon\searrow 0}M^{\frac{r}{p}-1}[a(\varepsilon)M]^{1-\frac{r}{p}}\sum_{n\le [a(\varepsilon)M]} n^{\frac{r}{p}-2}I_1\\
=\ &0.
\end{aligned}$$

Now, we estimate I_2.
By Lemma 3, note that for $1<\alpha<2$, $E|\frac{|S_n|}{\sqrt{n}}|^\alpha<\infty$, and by Markov's inequality, for $x\ge n^{\frac{1}{p}-\frac{1}{2}}$,

$$\begin{aligned}
I_2 &= \int_{n^{\frac{1}{p}-\frac{1}{2}}}^{\infty}|P(\frac{|S_n|}{\sqrt{n}}\ge \varepsilon n^{\frac{1}{p}-\frac{1}{2}}+x)-P(|N|\ge \varepsilon n^{\frac{1}{p}-\frac{1}{2}}+x)|dx\\
&\le \int_{n^{\frac{1}{p}-\frac{1}{2}}}^{\infty} cx^{-\alpha}dx\\
&\le cn^{(\frac{1}{p}-\frac{1}{2})(1-\alpha)}\longrightarrow 0 \ \text{ as } \ n\to\infty, \ \text{for} \ \ 1<\alpha<2, \quad 1\le p<r<2.
\end{aligned}$$

So, we get

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\varepsilon^{\frac{2(r-p)}{2-p}}\sum_{n\le a(\varepsilon)M} n^{\frac{r}{p}-2}I_2\\
=\ &\lim_{\varepsilon\searrow 0}\varepsilon^{\frac{2(r-p)}{2-p}}[a(\varepsilon)M]^{\frac{r}{p}-1}[a(\varepsilon)M]^{1-\frac{r}{p}}\sum_{n\le a(\varepsilon)M} n^{\frac{r}{p}-2}I_2\\
\le\ &\lim_{\varepsilon\searrow 0}M^{\frac{r}{p}-1}[a(\varepsilon)M]^{1-\frac{r}{p}}\sum_{n\le [a(\varepsilon)M]} n^{\frac{r}{p}-2}I_2\\
=\ &0.
\end{aligned}$$

Thus, $\lim_{\varepsilon\searrow 0}\varepsilon^{\frac{2(r-p)}{2-p}}\sum_{n\le a(\varepsilon)M} n^{\frac{r}{p}-2}(I_1+I_2)=0.$

Proposition 3. For any $1\le p<r<2$ and $M>1$, uniformly with respect to $\varepsilon>0$, we have

$$\lim_{M\to\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{\frac{2(r-p)}{2-p}}\sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2}E((|S_n|-\varepsilon n^{\frac{1}{p}})-E(|N|-\varepsilon n^{\frac{1}{p}-\frac{1}{2}}))=0.$$

Proof. It suffices to show that

$$\lim_{M\to\infty} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2} E(|N| - \varepsilon n^{\frac{1}{p}-\frac{1}{2}}) = 0 \tag{3.1}$$

and

$$\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2} E(|S_n| - \varepsilon n^{\frac{1}{p}}) = 0. \tag{3.2}$$

First, we estimate that $\lim_{M\to\infty} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2} E(|N| - \varepsilon n^{\frac{1}{p}-\frac{1}{2}}) = 0.$

$$\begin{aligned}
&\lim_{M\to\infty} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2} E(|N| - \varepsilon n^{\frac{1}{p}-\frac{1}{2}}) \\
\le\ & C \lim_{M\to\infty} \varepsilon^{\frac{2(r-p)}{2-p}} \int_{a(\varepsilon)M} x^{\frac{r}{p}-2} P(|N| \ge \varepsilon x^{\frac{1}{p}-\frac{1}{2}}) dx \\
\le\ & C \lim_{M\to\infty} \varepsilon^{\frac{2(r-p)}{2-p}} \int_{\varepsilon(a(\varepsilon)M)\frac{2-p}{2p}} \left(\frac{t}{\varepsilon}\right)^{\frac{2p}{2-p}(\frac{r}{p}-2)} P(|N| \ge t) \frac{2p}{2-p} (t)^{\frac{2p}{2-p}-1} (\frac{1}{\varepsilon})^{\frac{2p}{2-p}} dt
\end{aligned}$$

by letting $t = \varepsilon x^{\frac{1}{p}-\frac{1}{2}}$

$$= \ C \frac{2p}{2-p} \lim_{M\to\infty} \int_{M^{\frac{1}{p}-\frac{1}{2}}}^{\infty} t^{\frac{2(r-p)}{2-p}-1} P(|N| \ge t) dt \ \longrightarrow \ 0$$

as uniformly in $0 < \varepsilon < 1$. Thus (3.1) is now proved.
Note that $a_{ni} = {a_{ni}}^+ - {a_{ni}}^-$, where ${a_{ni}}^+ = \max(a_{ni},\ 0)$ and ${a_{ni}}^- = \max(-a_{ni},\ 0)$.
Secondly, it suffices to show that for every $\varepsilon > 0$,

$$\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2} E(|\sum_{i=-\infty}^{\infty} |{a_{ni}}^+ \varepsilon_i| - \varepsilon n^{\frac{1}{p}}) = 0 \tag{3.3}$$

and

$$\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2} E(|\sum_{i=-\infty}^{\infty} {a_{ni}}^- \varepsilon_i| - \varepsilon n^{\frac{1}{p}}) = 0. \tag{3.4}$$

We prove only (3.3), the proof of (3.4) is analogous.
Consider the decomposition

$$\varepsilon_i^{'} = {a_{ni}}^+ \varepsilon_i I(|{a_{ni}}^+ \varepsilon_i| \le x) + xI({a_{ni}}^+ \varepsilon_i > x) \ - \ xI({a_{ni}}^+ \varepsilon_i < -x).$$

Then

$$\begin{aligned}
&\lim_{M\to\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{\frac{2(r-p)}{2-p}}\sum_{n>a(\varepsilon)M}n^{\frac{r}{p}-2}\int_{\varepsilon n^{\frac{1}{p}}}^{\infty}P(|\sum_{i=-\infty}^{\infty}{a_{ni}}^{+}\varepsilon_i|>x)dx\\
\leq\ &\lim_{M\to\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{\frac{2(r-p)}{2-p}}\sum_{n>a(\varepsilon)M}n^{\frac{r}{p}-2}\int_{\varepsilon n^{\frac{1}{p}}}^{\infty}P(|\sum_{i=-\infty}^{\infty}\varepsilon_i^{'}|>\frac{x}{2})dx\\
+\ &\lim_{M\to\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{\frac{2(r-p)}{2-p}}\sum_{n>a(\varepsilon)M}n^{\frac{r}{p}-2}\int_{\varepsilon n^{\frac{1}{p}}}^{\infty}P(\sum_{i=-\infty}^{\infty}|{a_{ni}}^{+}\varepsilon_i|>x)dx\\
=\ &\lim_{M\to\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{\frac{2(r-p)}{2-p}}\sum_{n>a(\varepsilon)M}n^{\frac{r}{p}-2}[\int_{\varepsilon n^{\frac{1}{p}}}^{\infty}P(|\sum_{i=-\infty}^{\infty}\varepsilon_i^{'}|>\frac{x}{2})dx\\
&\qquad\qquad+\int_{\varepsilon n^{\frac{1}{p}}}^{\infty}P(\sum_{i=-\infty}^{\infty}|{a_{ni}}^{+}\varepsilon_i|>x)dx]\\
=:\ &I_1+I_2.
\end{aligned}$$

Next, we observe that

$$\sum_{k=1}^{n}X_k=\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty}a_{k+i}\varepsilon_i,\ \text{set}\ \ a_{ni}=\sum_{k=1}^{n}a_{k+i}.$$

Then

$$\sum_{k=1}^{n}X_k=\sum_{i=-\infty}^{\infty}a_{ni}\varepsilon_i$$

From Lemma 1, we can assume, without loss of generality, that

$$\sum_{i=-\infty}^{\infty}{a_{ni}}^{+}\leq n,\ n\geq 1\text{ and }{a_{ni}}^{+}\leq 1$$

According to Markov's inequality and Lemma 2, we have

$$\begin{aligned}
I_1 &= \sum_{n>a(\varepsilon)M}n^{\frac{r}{p}-2}\int_{\varepsilon n^{\frac{1}{p}}}^{\infty}P(|\sum_{i=-\infty}^{\infty}\varepsilon_i^{'}|>\frac{x}{2})dx\\
&\leq C\sum_{n>a(\varepsilon)M}n^{\frac{r}{p}-2}\int_{\varepsilon n^{\frac{1}{p}}}^{\infty}x^{-2}\sum_{i=-\infty}^{\infty}E|\varepsilon_i^{'}|^2dx\\
&\leq C\sum_{n>a(\varepsilon)M}n^{\frac{r}{p}-2}\int_{\varepsilon n^{\frac{1}{p}}}^{\infty}(x^{-2}\sum_{i=-\infty}^{\infty}E|{a_{ni}}^{+}\varepsilon_i|^2I(|a_{ni}\varepsilon_i|>x)\\
&\qquad\qquad+\sum_{i=-\infty}^{\infty}P(|{a_{ni}}^{+}\varepsilon_i|>x))dx\\
&=:\ I_3+I_4
\end{aligned}$$

Set $I_{nj} = \{i \in N | (j+1)^{\frac{-1}{p}} < |a_{ni}{}^{+}| \le j^{\frac{-1}{p}}\},\ j = 1, 2, \cdots$. Then $\bigcup_{j\ge 1} I_{nj} = N$, it is easy to verify from Lemma 1 that

$$\sum_{j=1}^{k} I_{nj} \le Cn(k+1)^{\frac{1}{p}}. \tag{3.5}$$

For I_3, using (3.5), we have

$$\begin{aligned}
I_3 &\le C \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} x^{-2} \sum_{i=-\infty}^{\infty} j^{\frac{-2}{p}} E|\varepsilon_1|^2 I(|\varepsilon_1| \ge x^{\frac{1}{p}}) dx \\
&\le C \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} x^{-2} \sum_{j=1}^{\infty} (\# I_{nj}) j^{\frac{-2}{p}} \sum_{k \ge j x^p} E|\varepsilon_1|^2 I(k \le |\varepsilon_1|^p < k+1) dx \\
&\le C \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} x^{-2} \sum_{k=[x^p]}^{\infty} \sum_{j=1}^{[\frac{k}{x^p}]} (\# I_{nj}) j^{\frac{-2}{p}} E|\varepsilon_1|^2 I(k \le |\varepsilon_1|^p < k+1) dx \\
&\le C \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} x^{-2} \sum_{k=[x^p]}^{\infty} \sum_{j=1}^{[\frac{k}{x^p}]} (\# I_{nj}) E|\varepsilon_1|^2 I(k \le |\varepsilon_1|^p < k+1) dx \\
&\le C \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} x^{-2} \sum_{k=[x^p]}^{\infty} n(\frac{k}{x^p}+1)^{\frac{1}{p}} E|\varepsilon_1|^2 I(k \le |\varepsilon_1|^p < k+1) dx \\
&\le C \int_{a(\varepsilon)M}^{\infty} y^{\frac{r}{p}-2} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} x^{-2} \sum_{k=[x^p]} y(\frac{k}{x^p}+1)^{\frac{1}{p}} E|\varepsilon_1|^2 I(k \le |\varepsilon_1|^p < k+1) dx dy \\
&\le C \int_{a(\varepsilon)M}^{\infty} y^{\frac{r}{p}-1} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} x^{-3} \sum_{k=[x^p]} k^{\frac{1}{p}} E|\varepsilon_1|^2 I(k \le |\varepsilon_1|^p < k+1) dx dy \\
&\le C\varepsilon^{-r} \int_{\varepsilon(a(\varepsilon)M)^{\frac{1}{p}}}^{\infty} t^{r-1} \int_{t}^{\infty} x^{-3} \sum_{k=[x^p]} k^{\frac{1}{p}} E|\varepsilon_1|^2 I(k \le |\varepsilon_1|^p < k+1) dx dy
\end{aligned}$$

by letting $t = \varepsilon y^{\frac{1}{p}}$

$$\begin{aligned}
&\le C\varepsilon^{-r} \int_{\varepsilon^{\frac{-p}{2-p}} M^{\frac{1}{p}}}^{\infty} \left(\int_{\varepsilon^{\frac{-p}{2-p}} M^{\frac{1}{p}}}^{x} t^{r-1} dt \right) x^{-3} \sum_{k=[x^p]} k^{\frac{1}{p}} E|\varepsilon_1|^2 I(k \le |\varepsilon_1|^p < k+1) dx \\
&\le C\varepsilon^{-r} \sum_{k=[\varepsilon^{\frac{-p^2}{2-p}} M]} k^{\frac{1}{p}} E|\varepsilon_1|^2 I(k \le |\varepsilon_1|^p < k+1) \int_{\varepsilon^{\frac{-p}{2-p}} M^{\frac{1}{p}}}^{(k+1)^{\frac{1}{p}}} x^{r-3} dx. \\
&\le C\varepsilon^{-r} \sum_{k=[\varepsilon^{\frac{-p^2}{2-p}} M]} k^{\frac{1}{p}} E|\varepsilon_1|^2 I(k \le |\varepsilon_1|^p < k+1)(k+1)^{\frac{r-2}{p}}
\end{aligned}$$

$$\leq \; C\varepsilon^{-r} \sum_{k=[\varepsilon^{\frac{-p^2}{2-p}}M]} k^{\frac{1}{p}} E|\varepsilon_1|^2 I(k \leq |\varepsilon_1|^p < k+1)\varepsilon^{\frac{-p(r-2)}{2-p}} M^{\frac{r-2}{p}}$$

$$\leq \; C\varepsilon^{-r+\frac{-p(r-2)}{2-p}} M^{\frac{r-2}{p}} E|\varepsilon_1|^3 I(|\varepsilon_1| \geq \varepsilon^{\frac{-p^2}{2-p}} M^{\frac{1}{p}})$$

By $E|\varepsilon_1|^3 < \infty$, it follows that, for $1 \leq p < r < 2$, we have

$$\lim_{M\to\infty} \lim_{\varepsilon\searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} I_3 \; \leq C \lim_{M\to\infty} M^{\frac{r-2}{p}} = 0 \tag{3.6}$$

For I_4, we have

$$\begin{aligned} I_4 &= C \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} \sum_{i=-\infty}^{\infty} P(|a_{ni}{}^{+}\varepsilon_i| \geq x)dx \\ &\leq C \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-2} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} \frac{\sum_{i=-\infty}^{\infty} a_{ni}{}^2 E\varepsilon_i^2}{x^2} dx \\ &\leq C \sum_{n>a(\varepsilon)M} n^{\frac{r}{p}-1} (\varepsilon n^{\frac{1}{p}})^{-1} \\ &= C \sum_{n>a(\varepsilon)M} \varepsilon^{-1} n^{\frac{r}{p}-1-\frac{1}{p}} \\ &\leq C\varepsilon^{-1}(a(\varepsilon)M)^{\frac{r}{p}-\frac{1}{p}} \\ &= C\varepsilon^{\frac{-2r+p}{2-p}} M^{\frac{r-1}{p}} \end{aligned}$$

Since $1 \leq p < r < 2$, we can conclude that $\frac{-p}{2-p} < 0$ and $\frac{r-1}{p} > 0$. Thus we have

$$\begin{aligned} &\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} I_4 \\ \leq \; &\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{\frac{-p}{2-p}} M^{\frac{r-1}{p}} \\ = \; &0. \end{aligned} \tag{3.7}$$

Therefore combining (3.6) and (3.7), (3.2) is now proved.

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