Derivative Approximation by Interpolation
Polynomials of Bernstein Type of the Third Kind

Xuegang Yuan

School of Science, Dalian Nationalities University
Dalian 116600, Liaoning, P.R. China
yxg1971@163.com

Abstract

In this paper, we mainly study the problems of derivative approximation of interpolation polynomials. Based on the zero nodes of the Chebyshev polynomials of the second kind, we construct an interpolation polynomial of Bernstein type of the third kind. It is proved that the derivative of the constructed polynomial converges uniformly to the first-order derivative of the interpolated function for any \(x \in (-1, 1)\) and that the best approximation order is also present.

Mathematics Subject Classification: 41A28

Keywords: interpolation polynomial of Bernstein type; derivative approximation; uniform convergence; the best approximation order

1. Formulation and Results

Let \(f(x)\) be any functions with the first-order continuous derivative for any \(x \in [-1, 1]\), namely, \(f(x) \in C_{[-1,1]}^1\), and let \(U_n(x)\) be the Chebyshev polynomials of the second kind, where

\[
U_n(x) = \frac{\sin(n + 1)\theta}{\sin \theta}, \quad (x = \cos \theta).
\]

The zero nodes of the orthogonal polynomial \((1 - x^2)U_n(x)\) with the weight function \(\rho(x) = \sqrt{1 - x^2}\) are given by

\[
x_k = \cos \theta_k = \cos \frac{k\pi}{n + 1}, \quad (k = 0, 1, \ldots, n + 1).
\]
The Lagrange interpolation polynomial of \( f(x) \in C_{[-1,1]} \), which is equal to \( f(x_k) \) at the interpolation nodes \( x_k \) \((k = 0, 1, \cdots, n + 1)\), is given by

\[
L_n(f; x) = \sum_{k=0}^{n+1} f(x_k) \mu_k(x),
\]

where

\[
\mu_k(x) = (-1)^{k+1} \frac{(1-x^2)U_n(x)}{(n+1)(x-x_k)}, \quad k = 0, 1, 2, \cdots, n + 1
\]

are the basic Lagrange interpolation functions based on the interpolation nodes \( x_k \).

As is well known, the Lagrange interpolation polynomials do not converge uniformly to arbitrary continuous functions\(^1\). To improve the uniform convergence of the known Lagrange interpolation polynomials, we construct a class of interpolation polynomials \( B_n(f; x) \) based on the idea put forward by S.N. Bernstein\(^2\), moreover, we prove that the first-order derivative of the interpolation polynomial converges uniformly to the first-order derivative of the interpolated function \( f'(x) \) and that the approximation order is the best. \( B_n(f; x) \) is constructed as follows.

For the given nature number \( l > 2 \), the zero nodes, \( x_1, x_2, \cdots, x_n \) except \( x_0 \) and \( x_{n+1} \), are divided into \( s \) groups according to \( 2l \), namely, \( n = 2ls + r \), \((0 \leq r < 2l)\). At the \((2lt - 1) - th\) nodes and the \(2lt - th\) nodes, \((t = 1, 2, \cdots, s)\), let

\[
D_{2lt-1} = f(x_{2lt-1}) + \frac{1}{2} \sum_{p=1}^{l} (f(x_{2lt-1+2p-2}) - f(x_{2lt-1+2p-1}))
\]

\[
+ \frac{1}{4} (f(x_{2lt}) - f(x_{2lt-1})) = f(x_{2lt-1}) + D_{2lt-1}^*,
\]

\[
D_{2lt} = f(x_{2lt}) + \frac{1}{2} \sum_{p=1}^{l} (f(x_{2lt+2p-1}) - f(x_{2lt+2p}))
\]

\[
+ \frac{1}{4} (f(x_{2lt+1}) - f(x_{2lt+1})) = f(x_{2lt}) + D_{2lt}^*,
\]

then the expression of the interpolation polynomial \( B_n(f; x) \) is denoted by

\[
B_n(f; x) = \sum_{k=0}^{n+1} D_k \mu_k(x),
\]

where \( D_k \) is given by Eq.\((4)\) as \( k = 2lt - 1(t = 1, 2, \cdots, s)\) and is given by Eq.\((5)\) as \( k = 2lt \), otherwise, \( D_k = f(x_k) \).

For \( B_n(f; x) \), we have the following results:

**Theorem 1** If \( f(x) \in C_{[-1,1]/}, \) \((0 \leq j \leq 3)\), then for any \( x \in [-1, 1] \) we have

\[
|B_n(f; x) - f(x)| = O\left(\frac{1}{n^j} \omega_{(f)}^{(j)} \frac{1}{n} + \frac{1}{n^{j+1}}\right),
\]

where \( \omega_{(f)}^{(j)} \) is the modulus of continuity of the \( j \)-th derivative of \( f \).
Theorem 2 If \( f(x) \) is an arbitrary continuous function on \([-1, 1]\) and has the first-order continuous derivative in \((-1, 1)\), then we have

\[
|B'_n(f; x) - f'(x)| = O \left( \frac{1}{\sqrt{1 - x^2}} \left( \omega(f', \frac{1}{n}) + \frac{\|f'\|}{n} \right) \right), \tag{8}
\]

where \( O \) is independent of \( n, x, f, f', \cdots, f^{(j)} \), \( \omega(f^{(j)}, x) \) is the modulus of continuity of \( f^{(j)}(x) \) and \( \|f'\| = \max_{-1 < x < 1} |f'(x)| \).

2. Proof of Theorem

From one of the properties of Lagrange interpolation polynomial, i.e., \( \sum_{k=0}^{n+1} \mu_k(x) = 1 \) we have the following expression:

\[
B_n(f; x) - f(x) = \sum_{k=0}^{n+1} (f(x_k) - f(x))\mu_k(x) + \sum_{t=1}^{s} [D^s_{2t-1} \mu_{2t-1}(x) + D^s_{2t} \mu_{2t}(x)]
\]

\[
= \sum_{k=0}^{n+1} (f(x_k) - f(x))q_k(x) + \frac{1}{4} (y_0 - y_1)\mu_1(x) + \frac{1}{4} (y_{n+1} - y_n)\mu_{n+1}(x) +
\]

\[
\frac{1}{4} \sum_{t=1}^{s} \sum_{p=1}^{l-1} \sum_{i=1}^{n+1} (f(x_{j-1}) - 2f(x_j) + f(x_{j+1}))(\mu_i(x) - \mu_j(x)) +
\]

\[
\frac{1}{4} \sum_{t=1}^{s} \sum_{p=1}^{l-1} \sum_{i=1}^{n+1} (f(x_{j}) - 2f(x_{j+1}) + f(x_{j+2}))(\mu_{i+1}(x) - \mu_{j+1}(x)) -
\]

\[
\frac{1}{4} \sum_{p=2s+2}^{n} \sum_{i=1}^{s} (f(x_{p-1}) - 2f(x_p) + f(x_{p+1}))\mu_p(x)
\]

\[
= \sum_{v=1}^{6} A_v, \tag{9}
\]

where \( i = 2lt, j = 2l(t - 1) + 2p \) and \( q_0(x) = \frac{1}{4} (3\mu_0(x) + \mu_1(x)) \), \( q_k(x) = \frac{1}{4} (\mu_{k-1}(x) + 2\mu_k(x) + \mu_{k+1}(x)) \), \( q_{n+1}(x) = \frac{1}{4} (\mu_n(x) + 3\mu_{n+1}(x)) \).

Using the similar method in [3], we can prove that Theorem 1 is valid without any difficulty, and thus we omit the proof of Theorem 1.

Differentiating both sides of Eq.(9) with respect to \( x \), we have

\[
B'_n(f; x) - f'(x) = \sum_{v=1}^{6} A'_v. \tag{10}
\]

Next we respectively estimate \( A'_v(i = 1, 2, \cdots, 6) \), as follows.
(i) Estimation of $A'_1$. It is not difficult to show that the following expressions are valid
\[ \sum_{k=0}^{n+1} q_k(x) = 1, |x - x_k| = O \left( \sin \frac{\theta - \theta_k}{2} \right), |(x - x_k)q_k(x)| = O \left( \frac{1}{\sqrt{1 - x^2}} \right). \]

Using the similar method in [4], we have
\[ |A'_1| = O \left( \frac{1}{\sqrt[4]{1 - x^2}} \left( \frac{1}{n^2} + \frac{\| f' \|}{n} \right) \right). \tag{11} \]

(ii) Estimations of $A'_2$ and $A'_3$. It is not difficult to show that $\mu'_k(x) = O \left( \frac{n}{\sqrt{1 - x^2}} \right), (k = 0, 1, \cdots, n + 1)$ by using $\mu_k(x) = O(1)$ and the known Bernstein inequality. Moreover, using the known Lagrange mean value theorem and $x_k - x_{k+1} = O \left( \frac{1}{n^2} \right)$, we have
\[ |A'_v| = O \left( \frac{1}{n \sqrt{1 - x^2}} \left( \frac{1}{n^2} + \frac{\| f' \|}{n} \right) \right), (v = 2, 3). \tag{12} \]

(iii) Estimations of $A'_4, A'_5$ and $A'_6$. Again using the known Lagrange mean value theorem and $x_k - x_{k+1} = O \left( \frac{1}{n^2} \right)$, we have
\[ f(x_j-1) - 2f(x_j) + f(x_{j+1}) = O \left( \frac{1}{\sqrt[4]{1 - x^2}} \left( \frac{1}{n^2} + \frac{\| f' \|}{n} \right) \right). \]

On the other hand, using the properties of basic functions of Lagrange interpolation polynomial, namely, Eq.(3), we have
\[ |\mu_i(x) - \mu_j(x)| = O \left( \frac{n}{h^2 \sqrt{1 - x^2}} \right), |\mu_{i+1}(x) - \mu_{j+1}(x)| = O \left( \frac{n}{h^2 \sqrt{1 - x^2}} \right), \]
where $i = 2lt, j = 2l(t - 1) + 2p$ and $h = \min\{|k_0 - i|, |k_0 - j|\}$.

Consequently,
\[ |A'_v| = O \left( \frac{1}{\sqrt[4]{1 - x^2}} \left( \frac{1}{n^2} + \frac{\| f' \|}{n} \right) \right), (v = 4, 5, 6). \tag{13} \]

From Eqs.(11) (13), we know that the estimation (10) is valid.

References


Received: June 26, 2008