On a Reverse of Hardy-Hilbert's Integral Inequality via Minkowski's Inequality

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Abstract

The reverse of the Hardy-Hilbert's Integral inequality via Minkowski's inequality in its general form is given. As an application, a special case is deduced.

Keywords: Hardy-Hilbert's inequality, Minkowski's inequality

1. Introduction

Let $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(t) \, dt < \infty \quad \text{and} \quad 0 < \int_0^\infty g^2(t) \, dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x) g(y)}{x+y} \, dx \, dy < \pi \left( \int_0^\infty f^2(t) \, dt \int_0^\infty g^2(t) \, dt \right)^{1/2},$$

where the constant factor $\pi$ is the best possible (cf. Hardy et al. [2]). Inequality (1) is well known as Hilbert's integral inequality. This inequality had been extended by Hardy [1] as follows

If $p>1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t) \, dt < \infty \quad \text{and} \quad \int_0^\infty g^q(t) \, dt < \infty,$$

then
where the constant factor \( \frac{\pi}{\sin(\pi/p)} \) is the best possible. Inequality (2) is called Hardy-Hilbert’s integral inequality and is important in analysis and application (cf. Mitrinovic et al. [3]).

Gradually, B. Yang gave the following extensions of (2) as follows:

Theorem A[4]. If \( \lambda > 2 - \min\{p, q\} \), \( f, g \geq 0 \), satisfy
\[
0 < \int_0^\infty t^{1-\lambda} f^p(t) dt < \infty \text{ and } \int_0^\infty t^{1-\lambda} g^q(t) dt < \infty,
\]
then
\[
\left(\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^p} dx \, dy \right)^{1/p} < k_\lambda(p) \left( \int_0^\infty t^{1-\lambda} f^p(t) dt \right)^{1/p} \left( \int_0^\infty t^{1-\lambda} g^q(t) dt \right)^{1/q},
\]
where the constant factor \( k_\lambda(p) = B\left( \frac{\lambda p - 2}{p}, \frac{\lambda q - 2}{q} \right) \) is the best possible, \( B \) is the beta function.

Theorem B[5]. If \( n \in \mathbb{N} - \{1\} \), \( p_i > 1 \), \( \sum_{i=1}^n \frac{1}{p_i} = 1 \), \( \lambda > 0 \), \( f_i \geq 0 \) satisfy
\[
0 < \int_0^\infty t^{p_i-1-\lambda} f_i^{p_i}(t) dt < \infty \quad (i = 1, 2, ..., n),
\]
then
\[
\int_0^\infty \cdots \int_0^\infty \frac{1}{\left( \sum_{j=1}^n x_j \right)^{\lambda}} \prod_{i=1}^n f_i(x_i) dx_1 ... dx_n < \frac{1}{\Gamma \lambda} \prod_{i=1}^n \Gamma \left( \frac{\lambda}{p_i} \right) \left( \int_0^\infty t^{p_i-1-\lambda} f_i^{p_i}(t) dt \right)^{1/p_i},
\]
where the constant factor \( \frac{1}{\Gamma \lambda} \prod_{i=1}^n \Gamma \left( \frac{\lambda}{p_i} \right) \) is the best possible.

2. Lemmas

The following is needed for our aim
Lemma 2.1. Let \( f_i \geq 0 \), \( 0 \leq m \leq \frac{f_i^p}{\left( \sum_{i=1}^{n} f_i \right)^q} \leq M \), \( i = 1, \ldots, n \), \( p > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \).

Then

\[
\left( \int \left( \sum_{i=1}^{n} f_i \right)^p \right)^{\frac{1}{p}} \geq \left( \frac{m}{M} \right)^{\frac{p-1}{p^2}} \sum_{i=1}^{n} \left( \int f_i^p \right)^{\frac{1}{p}}.
\]

In particular

\[
\left( \int (f + g)^p \right)^{\frac{1}{p}} \geq \left( \frac{m}{M} \right)^{\frac{1}{pq}} \left( \left( \int f^p \right)^{\frac{1}{p}} + \left( \int g^p \right)^{\frac{1}{p}} \right).
\]

Proof. We have

\[
f_i^{\frac{p}{q}} \leq M^{\frac{1}{q}} F,
\]
tends to

\[
\int f_i^p = \int f_i^{\frac{p}{q} \cdot \frac{q}{p}} = \int f_i^{\frac{p}{q}} f_i^q \leq M^{\frac{1}{q}} \int f F,
\]
and hence

\[
\left( \int f_i^p \right)^{\frac{1}{p}} \leq M^{-\frac{1}{pq}} \left( \int f F \right)^{\frac{1}{p}},
\]

\[
\left( \int F^q \right)^{\frac{1}{q}} \leq m^{-\frac{1}{pq}} \left( \int f F \right)^{\frac{1}{q}}.
\]

Multiplying, we get

\[
\left( \int f_i^p \right)^{\frac{1}{p}} \left( \int F^q \right)^{\frac{1}{q}} \leq \left( \frac{m}{M} \right)^{\frac{1}{pq}} \left( \int f F \right).
\]

Now applying (7) with \( q = p/(p-1) \), we have

\[
\int \left( \sum_{i=1}^{n} f_i \right)^p = \int \left( \sum_{i=1}^{n} f_i \right) \left( \sum_{i=1}^{n} f_i \right)^{p-1} = \sum_{i=1}^{n} \int \left( f_i \left( \sum_{i=1}^{n} f_i \right)^{p-1} \right) \geq \left( \frac{m}{M} \right)^{\frac{p-1}{p^2}} \sum_{i=1}^{n} \left( \int f_i^p \right)^{\frac{1}{p}} \left( \int \left( \sum_{i=1}^{n} f_i \right)^p \right)^{\frac{p-1}{p}},
\]

which implies

\[
\left( \int \left( \sum_{i=1}^{n} f_i \right)^p \right)^{\frac{1}{p}} \geq \left( \frac{m}{M} \right)^{\frac{p-1}{p^2}} \sum_{i=1}^{n} \left( \int f_i^p \right)^{\frac{1}{p}}.
\]

Lemma 2.2. Let \( \lambda_j > 0 \), \( j = 1, \ldots, n \), \( \lambda = \sum_{j=1}^{n} \lambda_j \). Then
\[ \prod_{j=1}^{n} B \left( \lambda_j, \lambda - \frac{\sum_{j=1}^{n} \lambda_j}{n} \right) = \frac{1}{\Gamma(\lambda)} \prod_{j=1}^{n} \Gamma(\lambda_j), \quad i = 1, ..., n. \]

Proof. Follows just by opening the product.

### 3. Main Result

We aim to give the following

**Theorem 3.1.** Let \( F_i, f_i, f_i' > 0, \ f_i(0) = 0, \ f_i(\infty) = \infty, \ \lambda_i > 1, \ i = 1, ..., n, \)

\[ \lambda = \sum_{i=1}^{n} \lambda_i / p + 1 / q = 1, \ p > 1, \ m \leq \left( \sum_{j=1}^{n} f_j(x_j) \right)^{-1} \prod_{j=1}^{n} f_j^{\lambda_j - 1}(x_j) \]

\[ \sum_{i=1}^{n} \frac{F_i^p(x_i) f_i'(x_i)}{F_i(x_i) (f_i'(x_i))^{1/p}} \leq M. \]

Then

\[ \left( \int_{0}^{\infty} \sum_{i=1}^{n} F_i(x_i) (f_i'(x_i))^{1/p} \prod_{j=1}^{n} f_j(x_j)^{\lambda_j - 1} dx_i \right)^{1/p} \]

\[ \geq \left( \frac{m}{M} \right)^{p-1} \left( \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma(\lambda_i) \right)^{1/p} \sum_{i=1}^{n} \left( \int_{0}^{\infty} f_i^{\lambda_i - 1}(x_i) f_i'(x_i) F_i^p(x_i) dx_i \right)^{1/p} \]

provided the integrals on the right do exist.

Proof. On putting \( f_i(x_i) = \frac{F_i(x_i) (f_i'(x_i))^{1/p} \prod_{j=1}^{n} f_j^{\lambda_j - 1}(x_j)}{\left( \sum_{i=1}^{n} f_i(x_i) \right)^{1/p}} \) in (5), we obtain via Lemma 2.2,
Hardy-Hilbert's integral inequality

\[
\left( \int_0^\infty \cdots \int_0^\infty \frac{F_i(x_j) \left( f'(x_j) \right)^{1/p} \prod_{j=1}^n f(x_j)^{1/(x_j - 1)} \left( \sum_{j=1}^n f_j(x_j) \right)^\lambda}{\left( \sum_{j=1}^n f_j(x_j) \right)^\lambda} \, dx_1 \cdots dx_n \right)^{1/p}
\]

\[\geq \left( \frac{m}{M} \right)^{p-1} \sum_{i=1}^n \left( \int_0^\infty \cdots \int_0^\infty \frac{F_i^{\lambda}(x_j) \left( f_i'(x_j) \right)^{1/p} \prod_{j=1}^n f_j^{\lambda-1}(x_j) \left( \sum_{j=1}^n f_j(x_j) \right)^\lambda}{\left( \sum_{j=1}^n f_j(x_j) \right)^\lambda} \, dx_1 \cdots dx_n \right)^{1/p}
\]

\[= \left( \frac{m}{M} \right)^{p-1} \sum_{i=1}^n L_i^{1/p}(x_i).
\]

Now,

\[L_i(x_i) = \int_0^\infty \cdots \int_0^\infty \frac{F_i^{\lambda}(x_i) \left( f_i'(x_i) \right)^{1/p} \prod_{j=1}^n f_j^{\lambda-1}(x_j) \left( \sum_{j=1}^n f_j(x_j) \right)^\lambda}{\left( \sum_{j=1}^n f_j(x_j) \right)^\lambda} \, dx_1 \cdots dx_n
\]

\[= \int_0^\infty F_i^{\lambda}(x_i) \left( f_i'(x_i) \right)^{1/p} \prod_{j=1}^n f_j^{\lambda-1}(x_j) \, dx_i \left[ \int_0^\infty f_i^{\lambda-1}(x) \left( f_i'(x) \right) \, dx_1 \cdots \int_0^\infty f_{i-1}^{\lambda-1}(x_{i-1}) f_i'(x_{i-1}) \, dx_{i-1} \cdots \int_0^\infty f_0^{n-1}(x_0) \, dx_0 \right]
\]

\[\times \left[ \int_0^\infty \sum_{j=1}^{n-1} f_j(x_j) \left( \sum_{j=1}^{n-1} f_j(x_j) \right)^{\lambda-1} \, dx_j \right]
\]
As the last integral is equal to \( \int_0^\infty \frac{z^{\lambda_n-1}}{(1+z)\lambda} \, dz = B(\lambda_n, \lambda - \lambda_n) \), we have

\[
L_i(x_i) = B(\lambda_n, \lambda - \lambda_n) \int_0^\infty F_i^\prime(x_i) f_i(x_i) \left[ \sum_{j=1}^{n-1} f_j(x_j) \right] \, dx_i \int_0^\infty f_i(x_i) f_i'(x_i) \, dx_1, \ldots
\]

\[
\times \int_0^\infty f_i^{\lambda_i-1}(x_i) f_i'(x_i) \, dx_i \int_0^\infty f_i^{\lambda_{i+1}-1}(x_{i+1}) f_i'(x_{i+1}) \, dx_{i+1}, \ldots
\]

\[
\times \int_0^\infty f_n^{\lambda_n-1}(x_n) f_n'(x_n) \, dx_n \left( f(x) + \sum_{j=1}^{n-1} f_j(x_j) \right)^{\lambda - \lambda_n},
\]

Proceeding in this manner, obtaining

\[
L_i(x_i) = \prod_{j=2}^n B \left( \lambda_j, \lambda - \sum_{j=2}^n \lambda_j \right) \int_0^\infty F_i^\prime(x_i) f_i^{\lambda_i-1}(x_i) f_i'(x_i) \, dx_i \]

\[
\times \int_0^\infty f_i^{\lambda_i-1}(x_i) f_i'(x_i) \, dx_i \left( f_i(x_i) + f_i'(x_i) \right)^{\lambda - \lambda_n}, \ldots
\]

\[
= \prod_{j=2}^n B \left( \lambda_j, \lambda - \sum_{j=2}^n \lambda_j \right) \int_0^\infty F_i^\prime(x_i) f_i^{\lambda_i-1}(x_i) f_i'(x_i) \, dx_i
\]
Hardy-Hilbert's integral inequality

\[
\begin{align*}
&\times \int_0^\infty \frac{(f_i(x_i))^{\lambda_i-1} f_i'(x_i)}{f_i(x_i)} \frac{f_i'(x_i)}{f_i(x_i)} \left(1 + \frac{f_i(x_i)}{f_i(x_i)}\right)^{\lambda_i-\lambda_{i+1}-\lambda_{i+2}-\cdots-\lambda_1} \, dx_i \\
&= \prod_{j=1}^{n} B\left(\lambda_j, \lambda - \sum_{j=1}^{n} \lambda_j\right) \int_0^\infty F_i^p(x_i) f_i^{\lambda_i-1}(x_i) f_i'(x_i) \, dx_i ,
\end{align*}
\]

by applying Lemma 2.2, as the last integral is equal to

\[
B\left(\lambda_1, \lambda - \lambda_n - \cdots - \lambda_{i+1} - \lambda_{i+2} - \cdots - \lambda_1\right).
\]

This completes the proof of the Theorem.

Corollary 3.2. Let \( F_i \geq 0, \lambda_i > 1, \lambda = \sum_{i=1}^{n} \lambda_i, \ p > 1, \ 1/p + 1/q = 1, \)

\[
m \leq \frac{\sum_{j=0}^{n} x_j}{\prod_{j=1}^{n} x_j^{\lambda_i-1}} \frac{F_i^p(x_i)}{\sum_{i=1}^{n} f_i(x_i)} \leq M . \text{ Then}
\]

\[
\int_0^\infty \left(\sum_{i=1}^{n} \frac{f_i(x_i) \prod_{j=1}^{n} x_j^{\frac{1}{\lambda_j-1}}}{\left(\sum_{j=1}^{n} x_j\right)^{\lambda_i/p}} \, dx_1 \ldots dx_n\right)^p
\]

\[
\geq \frac{m^{p-1}}{M} \left(\frac{1}{\Gamma(\lambda) \prod_{j=1}^{n} \Gamma(\lambda_j)}\right)^{1/p} \sum_{i=1}^{n} \left(\int_0^\infty x_i^{\lambda_i-1} f_i^p(x_i) \, dx_i\right)^{1/p},
\]

\[
\text{provided the integrals on the right do exist} .
\]

Proof. Follows from Theorem 3.1 by putting \( F_i(x) = f_i(x), \ f_i(x) = x, \ i = 1, \ldots, n . \)
References


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