Finite Dimensional Approximations for Mixed Variational Inequalities with $J$-Pseudomonotone Operators

A. M. Saddeek and Sayed A. Ahmed

Abstract

In this paper, we construct and study the convergence of the finite dimensional approximation for the mixed variational inequalities with $J$-pseudomonotone operators and convex nondifferentiable functionals in real uniformly smooth Banach spaces which admit a weakly sequentially continuous duality map.

Mathematics Subject Classification: 35J25, 35J70, 49J40, 90C33

Keywords: Uniformly smooth Banach space, mixed variational inequalities, approximation, finite-dimensional, duality map, $J$-pseudomonotone operator

1 Introduction and Preliminaries

Let $V$ be a real Banach space with uniformly convex dual space $V^*$. we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $V$ and $V^*$. The modulus of smoothness of $V$ is defined by:

$$\rho_V(\tau) = \sup \left\{ \frac{\|u + \tau v\| + \|u - \tau v\|}{2} - 1 : \|u\| = \|v\| = 1 \right\}; \ \tau > 0.$$
It is well known that \( \rho_V(\tau) \leq \tau \forall \tau > 0 \) (see e.g.\cite{[8]}). If \( \rho_V(\tau) > 0 \forall \tau > 0 \), then \( V \) is said to be smooth. If there exist a constant \( C > 0 \) and a real number \( 1 < q < \infty \), such that \( \rho_V(\tau) \leq C\tau^q \), then \( V \) is said to be \( q \)-uniformly smooth. The Banach space \( V \) is called uniformly smooth if \( \lim_{\tau \to 0} \frac{\rho_V(\tau)}{\tau} = 0 \).

Typical examples of such \( p \)-uniformly smooth spaces are the Lebesgue \( L_p \), the sequence \( l_p \) and the Sobolev \( W^{(s)}_p \) spaces for \( 1 < p \leq 2 \) (see e.g.\cite{[7]}). For a given gauge function \( \Phi(t) = t^{p-1} \), \( 1 < p < \infty \), this means for a mapping \( \Phi: \mathbb{R}^+ \to \mathbb{R}^+ \) which is continuous and strictly increasing with \( \Phi(0) = 0 \) and \( \lim_{t \to +\infty} \Phi(t) = +\infty \), the duality mapping \( J: V \to V^* \) with respect to \( \Phi \) is given by

\[
\langle u, Ju \rangle = \|u\|\|Ju\|, \quad \|Ju\| = \|u\|^{p-1}
\]

for all \( u \in V \). Such a mapping \( J \) is said to be weakly sequentially continuous if \( J \) is sequentially continuous relative to the weak topologies on both \( V \) and \( V^* \). The spaces \( l_p \), \( 1 < p < \infty \), posses duality mappings which are weakly sequentially continuous (see e.g.\cite{[2]}). In \cite{[6]} Opial, showed that no spaces \( L_p \), \( p > 1, p \neq 2 \), possesses a weakly sequentially continuous duality mapping. It is well known (see e.g.\cite{[8], [4]}) that \( J \) is single-valued odd, and is uniformly continuous on bounded sets if \( V^* \) is uniformly convex.

Therefore, we always suppose that \( V \) is a \( q \)-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and its dual space \( V^* \) is uniformly convex.

We always use the symbols \( \to \) and \( \rightharpoonup \) to indicate strong and weak convergence, respectively.

Let \( M \) be a nonempty closed convex set in \( V \) and \( F \) be a convex (not necessarily differentiable) functional on \( V \). Given \( A: V \to V \) nonlinear operator, we consider the problem of finding \( u \in M \) such that

\[
\langle Au, J(\eta - u) \rangle + F(\eta) - F(u) \geq \langle f, J(\eta - u) \rangle \quad \forall \eta \in M,
\]

where \( f \in V \) is a given element. The inequality of the type (1) is called the mixed variational inequality.

Throught the following, we assume that the variational inequality (1) has a solution \( u \in M \). To this end, it suffices, to assume that \( A \) is \( J \)-coercive, i.e.,

\[
\langle Au, Ju \rangle \geq \rho_1(\|u\|\|u\|^{p-1}, \quad p > 1, \quad \lim_{\xi \to +\infty} \rho_1(\xi) = +\infty;
\]

and \( J \)-pseudomonotone, i.e.,

\[
\text{if } u_n \rightharpoonup u \in M \text{ and } \lim_{n \to +\infty} \sup \langle Au_n, J(u_n - u) \rangle \leq 0
\]
imply
\[ \lim_{n \to +\infty} \inf \langle Au_n, J(u_n - v) \rangle \geq \langle Au, J(u - v) \rangle \quad \text{for all } v \in M, \]
and \( J \)-Lipschitz Continuous, i.e., there exists a constant \( L > 0 \) such that
\[ \|Au - Av\|^{q-1} \leq L^{q-1}\|J(u - v)\|, \quad q > 1, \]
and that the functional \( F \) satisfies the following condition:
\[ |F(u) - F(v)| \leq \gamma\|u - v\|^{q-1}, \quad \forall u, v \in V, \quad q > 1, \quad \gamma > 0. \quad (2) \]

We note that if \( V = H \) (\( J \equiv I \), the identity operator and \( q = 2 \)), then the above definition reduces to the standard definition of coercivity, pseudomonotonicity and Lipschitz continuity of the operator \( A \) (see e.g. [4]).

Consequently, the problem (1) becomes
\[ \text{find } u \in M \text{ such that, for all } \eta \in M \subset H \]
\[ \langle Au, \eta - u \rangle + F(\eta) - F(u) \geq \langle f, \eta - u \rangle \quad (3) \]
where \( f \in H \) is a given element.

This problem occurs, in particular, in the descriptions of stabilized filtration and equilibrium problems for soft shells (see [1], [5]).

The basic scope of this paper is to introduce the finite dimensional approximation and analyze its convergence for the mixed variational inequality (1).

## 2 An Internal Approximation

Let \( V_h \) be a finite dimensional subspace of the space \( V \). Suppose \( V_h \) is an internal approximation of \( V \):
\[ \forall v \in V, \text{ there exists } v_h \in V_h \text{ such that } v_h \to v \text{ when } h \to 0. \]

For each \( h \) we consider a nonempty closed convex subset \( M_h \subset V_h \) (observe that, in general, the set \( M_h \) is not a subset of \( M \)) approximating \( M \):
\[ \forall \eta \in M, \text{ there exists } \eta_h \in M_h \text{ such that } \eta_h \to \eta \text{ when } h \to 0, \quad (4) \]
and
\[ \text{if } \eta_h \in M_h, \quad \eta_h \to \eta \text{ in } V \text{ when } h \to 0, \text{ then } \eta \in M. \quad (5) \]
It is obvious that (5) is satisfied if $M_h \subset M$ for all $h$. Indeed, in this case \{\eta_h\} \subset M and $M$ is closed and convex (hence weakly closed), it follows that $\eta \in M$.

The approximate problem corresponding to problem (1) is the following problem:

$$\text{find } u_h \in M_h \text{ such that, for all } \eta_h \in M_h$$

$$\langle Au_h, J(\eta_h - u_h) \rangle + F(\eta_h) - F(u_h) \geq \langle f, J(\eta_h - u_h) \rangle. \quad (6)$$

Under the conditions imposed on $M_h, A,$ and $F$, there is a solution $u_h \in M_h$ to (6).

We now establish the convergence of the finite dimensional approximation in the following sense:

**Lemma 1** Let $F : V \to R^1$ be a convex continuous functional and let the operator $A : V \to V$ be $J$-coercive. If $u_h$ is the solution of problem (6), then

$$\|u_h\|_V \leq \tilde{c}, \quad (7)$$

where the constant $\tilde{c}$ is independent of $h$.

**Proof.** Since $F$ is convex and continuous, it is bounded below by a continuous affine function (see [3]) which can be written:

$$\langle f^*, J\eta \rangle - C^*, \quad \text{where } f^* \in V, \ \eta \in V \text{ and } C^* \in R^1. \quad (8)$$

Suppose that the assertion of the lemma is not true. In this case for each $N > 0$ we find $h_N$ such that $\|u_{h_N}\|_V \geq N$. Let $\alpha_N = \frac{1}{\|u_{h_N}\|_V}$. Clearly $\alpha_N \in (0, 1)$ for $N \geq 2$. We may assume without less of generality that $0 \in M_h$. Therefore $\alpha_N u_{h_N} \in M_h$. Putting $\eta_{h_N} = \alpha_N u_{h_N}$ in (6), using (8) and the fact that $J(\beta u) = \beta |\beta|^{p-2} J(u)$ $\forall u \in V, p > 1$, $\beta \in R^1$, we get

$$F(\eta_{h_N}) - \langle f, J\eta_{h_N} \rangle \geq \langle Au_{h_N}, J(u_{h_N} - \eta_{h_N}) \rangle + F(u_{h_N}) + \langle f, J(\eta_{h_N} - u_{h_N}) \rangle - \langle f, J\eta_{h_N} \rangle \geq \langle Au_{h_N}, J(u_{h_N} - \eta_{h_N}) \rangle + F(u_{h_N}) - \langle f, Ju_{h_N} \rangle \geq (1 - \alpha_N)|1 - \alpha_N|^{p-2}\langle Au_{h_N}, Ju_{h_N} \rangle + \langle f^*, Ju_{h_N} \rangle - C^* - \|f\|_V \|Ju_{h_N}\|_V. \geq (1 - \alpha_N)|1 - \alpha_N|^{p-2}\langle Au_{h_N}, Ju_{h_N} \rangle - (\|f^*\|_V + \|f\|_V) \|Ju_{h_N}\|_V - C^* = (1 - \alpha_N)|1 - \alpha_N|^{p-2}\langle Au_{h_N}, Ju_{h_N} \rangle - (\|f^*\|_V + \|f\|_V) \|u_{h_N}\|_V^{p-1} - C^*.$$

Further,

$$\|\eta_{h_N}\|_V = \alpha_N \|u_{h_N}\|_V = 1,$$
consequently, by the continuity of $F$ and the weak continuity of $J$, we have

$$F(\eta_N) - \langle f, J\eta_N \rangle \leq \tilde{C} < +\infty,$$

where the constant $\tilde{C}$ is independent of $h$.

On the other hand, by the $J$-coercivity of the operator $A$, for sufficiently large $N$ we obtain

$$(1 - \alpha_N)|1 - \alpha_N|^{p-2}(Au_{h_N}, Ju_{h_N}) - (\|f^*\|_V + \|f\|_V)u_{h_N}\|_V^{p-1} - C^*$$

$$\geq \frac{1}{2p-1}\rho_1(\|u_{h_N}\|_V) - (\|f^*\|_V + \|f\|_V)\|u_{h_N}\|_V^{p-1} - C^* \to +\infty$$

which is a contradiction and the assertion of the Lemma is proved.

**Theorem 1** If $u_h$ and $u$ denote the solutions of problems (6) and (1) respectively, there exists a subsequence $\{h_k\}_{k=1}^\infty, h_k \to 0$ as $k \to \infty$, such that $\{u_{h_k}\}_{k=1}^\infty$ converges weakly to $u$ in $V$ and any weak limit point $u_*$ of $\{u_h\}$ is a solution of problem (1). Moreover, if $\{u_{h_k}\}_{k=1}^\infty$ converges weakly to $u_*$ in $V$, then

$$\lim_{k \to \infty} \langle Au_{h_k} - Au_*, J(u_{h_k} - u_*) \rangle = 0. \quad (9)$$

**Proof.** From Lemma 1, it follows that $\{u_h\}$ is bounded and consequently there exists a subsequence $\{h_k\}_{k=1}^\infty, h_k \to 0$, as $k \to \infty$, such that $u_{h_k} \to u_* \in V$ as $k \to \infty$.

Let us show that $u_*$ is a solution of problem (1). By virtue of (5), we have $u_* \in M$. Let $\eta$ be an arbitrary element of $M$ and let $\eta_h \in M_h$ be constructed according to (4). In this case, the sequence $\{\eta_h\}$, converges strongly to $\eta$ in $V$ when $h \to 0$. Since $u_{h_k}$ is a solution of problem (6), then

$$\langle Au_{h_k}, J(\eta_{h_k} - u_{h_k}) \rangle + F(\eta_{h_k}) - F(u_{h_k}) \geq \langle f, J(\eta_{h_k} - u_{h_k}) \rangle,$$

consequently,

$$\langle Au_{h_k}, J(u_{h_k} - \eta) \rangle \leq \langle Au_{h_k}, J(u_{h_k} - \eta_h) \rangle + \langle Au_{h_k}, J(\eta_h - \eta) \rangle$$

$$\leq \langle Au_{h_k}, J(\eta_h - \eta) \rangle + F(\eta_h) - F(u_{h_k}) - \langle f, J(\eta_{h_k} - u_{h_k}) \rangle$$

$$\leq \|Au_{h_k}\|_V\|\eta_h - \eta\|_V^{p-1} + F(\eta_h) - F(u_{h_k}) - \langle f, J(\eta_{h_k} - u_{h_k}) \rangle$$

$$\leq \tilde{C}_1\|\eta_h - \eta\|_V^{p-1} + F(\eta_h) - F(u_{h_k}) - \langle f, J(\eta_{h_k} - u_{h_k}) \rangle.$$
On passing to the lim sup in the above formula we conclude

\[
\limsup_{k \to +\infty} \langle Au_{h_k}, J(u_{h_k} - \eta) \rangle \leq \tilde{C}_1 \limsup_{k \to +\infty} \|\eta_{h_k} - \eta\|_V^{p-1} + \limsup_{k \to +\infty} F(\eta_{h_k}) \\
- \liminf_{k \to +\infty} F(u_{h_k}) - \liminf_{k \to +\infty} \langle f, J(\eta_{h_k} - u_{h_k}) \rangle \\
\leq F(\eta) - F(u_\star) - \langle f, J(\eta - u_\star) \rangle,
\]

(since \(\eta_{h_k} \to \eta\), \(F\) is continuous as well weakly lower semicontinuous and since \(J\) is weakly continuous).

Thus,

\[
\limsup_{k \to +\infty} \langle Au_{h_k}, J(u_{h_k} - \eta) \rangle \leq F(\eta) - F(u_\star) - \langle f, J(\eta - u_\star) \rangle \quad \forall \eta \in M. \tag{10}
\]

Putting \(\eta = u_\star\) in (10), we obtain

\[
\limsup_{k \to +\infty} \langle Au_{h_k}, J(u_{h_k} - u_\star) \rangle \leq 0. \tag{11}
\]

From the \(J\)-pseudomonoticity of the operator \(A\), we have

\[
\langle Au_\star, J(u_\star - \eta) \rangle \leq \liminf_{k \to +\infty} \langle Au_{h_k}, J(u_{h_k} - \eta) \rangle \\
\leq F(\eta) - F(u_\star) - \langle f, J(\eta - u_\star) \rangle \quad \forall \eta \in M, \tag{12}
\]

that is \(u_\star\) is a solution of (1).

In a similar way, one can show that any weak limit point of the set \(\{u_h\}\) is a solution of the problem (1).

Finally, since \(u_{h_k} \rightharpoonup u_\star\) in \(V\) and \(J\) is weakly continuous, it follows that

\[
\lim_{k \to +\infty} \langle Au_\star, J(u_{h_k} - u_\star) \rangle = 0,
\]
therefore, from (11) and (12), we have

\[
0 \leq \liminf_{k \to +\infty} \langle Au_{h_k}, J(u_{h_k} - u_*) \rangle = \liminf_{k \to +\infty} \langle Au_{h_k}, J(u_{h_k} - u_*) \rangle \\
+ \lim_{k \to +\infty} \langle -Au_*, J(u_{h_k} - u_*) \rangle \\
\leq \liminf_{k \to +\infty} \langle Au_{h_k} - Au_*, J(u_{h_k} - u_*) \rangle \\
\leq \limsup_{k \to +\infty} \langle Au_{h_k} - Au_*, J(u_{h_k} - u_*) \rangle \\
\leq \limsup_{k \to +\infty} \langle Au_{h_k}, J(u_{h_k} - u_*) \rangle \\
+ \limsup_{k \to +\infty} \langle -Au_*, J(u_{h_k} - u_*) \rangle \\
= \limsup_{k \to +\infty} \langle Au_{h_k}, J(u_{h_k} - u_*) \rangle \\
\leq 0,
\]

that is, (9) holds and the theorem is proved.

References


A. M. Saddeek and Sayed A. Ahmed


Received: April 30, 2008