Applied Mathematical Sciences, Vol. 2, 2008, no. 59, 2915 - 2920

Weak Convergence Theorems of Three-Step Noor Iterative Scheme for I-quasi-nonexpansive Mappings in Banach Spaces*

Poom Kumam**

Department of Mathematics, Faculty of Science
King Mongkut's University of Technology Thonburi (KMUTT)
Bang-Mod, Bangkok 10140, Thailand
poom.kum@kmutt.ac.th

Warunya Kumethong

Department of Mathematics, Faculty of Science
King Mongkut's University of Technology Thonburi (KMUTT)
Bang-Mod, Bangkok 10140, Thailand
warunya_44@hotmail.com

Natchanok Jewwaiworn

Department of Mathematics, Faculty of Science
King Mongkut's University of Technology Thonburi (KMUTT)
Bang-Mod, Bangkok 10140. Thailand
lalabell1234@hotmail.com

Abstract

We establish the weak convergence of three-step Noor iterative scheme for an I-nonexpansive mappping in a Banach space which satisfies Opial's condition. The presented results in this paper, is connected with Kiziltunc and Ozdemir, "On convergence theorem for nonself I - nonexpansive mapping in Banach spaces", *Applied Mathematical Sciences*, Vol. 1, 2007, no. 48, 2379 – 2383 in [5], in sense of self maps and which extend and improve the recent ones announced by Rhoades and Temir, "Convergence theorems for I-nonexpansive mapping", *International Journal of Mathematical Sciences*, Volume 2006, Article ID 63435, Pages 1-4 in [12].

*Supported by the KMUTT Research Fund.
**Corresponding author

Mathematics Subject Classification : 47H09, 47H10, 47H20

Keywords: The three-step Noor iterative scheme, Opial's condition, nonexpansive mappings, I-quasi-nonexpansive mappings, fixed point

1. Introduction and preliminaries

Let X be a normed linear space, let K be a nonempty convex subset of X, and let $T : K \to K$ be a given mapping. The Mann iterative scheme $\{x_n\}$ is defined by $x_0 = x \in K$ and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n \tag{1.1}$$

The *Ishikawa iteration* defined by $x_0 = x \in K$ and

$$\left.\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \end{aligned}\right\} \tag{1.2}$$

The *Noor iteration* defined by $x_0 = x \in K$ and

$$\left.\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T x_n \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \end{aligned}\right\} \tag{1.3}$$

for every $n \in \mathrm{N}$, where α_n, β_n, γ_n is sequences in (0,1).

Recall that a Banach space X is said to satisfy Opial's condition [6] if, for each sequence $\{x_n\}$ in X, the condition $x_n \to x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \tag{1.4}$$

for all $y \in x$ with $y \neq x$. It is well known from [6] that all L_p spaces for $1 < p < \infty$ have this property. However, the L_p spaces do not, unless $p = 2$.

The following definitions and statements will be needed for the proof of our theorem.

Let K be a subset of a normed space $X = (x, \|.\|)$ and T and I self-mappings of K. Then T is called *I*-nonexpansive on K if

$$\|Tx - Ty\| \le \|Ix - Iy\| \tag{1.5}$$

for all $x, y \in K$ [9]. A mapping T is called *I*-quasi-nonexpansive on K if

$$\|Tx - f\| \le \|Ix - f\| \tag{1.6}$$

for all $x \in K$ and $f \in F(T) \cap F(I)$.

Let K be a closed convex bounded subset of uniformly convex Banach space $X = (X, \|\cdot\|)$ and T self-mappings of X .Then T is called nonexpansive on K if

$$\|Tx - Ty\| \le \|x - y\| \tag{1.7}$$

for all $x, y \in K$ .Let $F(T) = \{x \in K : Tx = x\}$ be denoted as the set of fixed points of a mapping T.

The first nonlinear ergodic theorem was proved by Baillon [1] for general nonexpansive mappings in Hilbert space H if K is a closed and convex subset of H and T has a fixed point, then for every $x \in K$, $\{T^n x\}$ is weakly almost convergent, as $n \to \infty$, to a fixed point of T. It was also shown by Pazy [7] that if H is a real Hilbert space and $(1/n \sum_{i=0}^{n-1} T^i x)$ converges weakly, as $n \to \infty$, to $y \in K$, then $y \in F(T)$. The concept of a quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. Dotson [2] studied quasi-nonexpansive mappings in Banach spaces. Recently, this concept was given by Kirk [4] in metric spaces which we adapt to a normed space as follows: T is called a quasi-nonexpansive mapping provided

$$\|Tx - f\| \le \|x - f\| \tag{1.8}$$

for all $x \in K$ and $f \in F(T)$.

Remark 1.1. From the above definitions it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and linear quasi-nonexpansive mapping are nonexpansive. But it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive.

There are many results on fixed points on nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, the strong and weak convergence of the sequence of certain iterates to a fixed point of quasi-nonexpansive maps was studied by Petryshyn and Williamson [8]. Their analysis was related to the convergence of Mann iterates studied by Dotson [2]. Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces was discussed by Ghosh and Debnath [3]. In [10], the weakly convergence theorem for *I*-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. In [11], convergence theorem of iterative schemes for nonexpansive mappings have been presented and generalized.

In [12], Rhoades and Temir considered T and I self-mappings of K , where T is an I -nonexpansive mapping. They established the weak convergence of the sequence of Mann iterates to a common fixed point of T and I . Later, Kiziltunc and Ozdemir [5], consider T and I nonself mappings of K , where T is an I-nonexpansive mappings. They proved the weak convergence of the sequence of modified Ishikawa iterates to a common fixed point of T and I .

In this paper, we consider T and I self-mappings of K, where T is an *I*-quasi-nonexpansive mapping. We establish the weak convergence of the sequence of Noor iterates (1.3) to a common fixed point of T and I.

2. The main result

In this section, we prove a weak convergence theorem which is our main result.

Theorem 2.1 Let K be a closed convex bounded subset of a uniformly convex Banach space X, which satisfies Opial's condition, and let T, I self-mappings of K with T be an I-quasi-nonexpansive mapping, I a nonexpansive on K. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of three-step Noor iterative scheme defined by (1.3) converges weakly to common fixed point of $F(T)\cap F(I)$.

Proof. If $F(T)\cap F(I)$ is nonempty and a singleton, then the proof is complete. We will assume that $F(T)\cap F(I)$ is nonempty and that $F(T)\cap F(I)$ is not a singleton. Let $f \in F(T)\cap F(I)$, we have

$$\begin{aligned} \|x_{n+1}-f\| &= \|(1-\alpha_n)x_n+\alpha_n Ty_n-(1-\alpha_n+\alpha_n)f\| \\ &= \|(1-\alpha_n)(x_n-f)+\alpha_n(Ty_n-f)\| \\ &\le (1-\alpha_n)\|x_n-f\|+\alpha_n\|Ty_n-f\| \\ &\le (1-\alpha_n)\|x_n-f\|+\alpha_n\|Iy_n-f\| \\ &\le (1-\alpha_n)\|x_n-f\|+\alpha_n\|y_n-f\| \qquad (2.1) \end{aligned}$$

and

$$\begin{aligned} \|y_n-f\| &= \|(1-\beta_n)(x_n-f)+\beta_n(Tz_n-f)\| \\ &\le (1-\beta_n)\|x_n-f\|+\beta_n\|Tz_n-f\| \\ &\le (1-\beta_n)\|x_n-f\|+\beta_n\|Iz_n-f\| \\ &\le (1-\beta_n)\|x_n-f\|+\beta_n\|z_n-f\| \qquad (2.2) \end{aligned}$$

and also, we get

$$\begin{aligned} \|z_n-f\| &= \|(1-\gamma_n)(x_n-f)+\gamma_n(Tx_n-f)\| \\ &\le (1-\gamma_n)\|x_n-f\|+\gamma_n\|Tx_n-f\| \\ &\le (1-\gamma_n)\|x_n-f\|+\gamma_n\|Ix_n-f\| \\ &\le (1-\gamma_n)\|x_n-f\|+\gamma_n\|x_n-f\| \qquad (2.3) \\ &= \|x_n-f\|. \end{aligned}$$

From (2.2) and (2.3), we obtain

$$\|y_n-f\| \le (1-\beta_n)\|x_n-f\|+\beta_n\|z_n-f\|$$

$$\leq (1-\beta_n)\|x_n - f\| + \beta_n\|x_n - f\|$$
$$= \|x_n - f\|. \qquad (2.4)$$

Substituting (2.4) in (2.1), we have

$$\begin{aligned}\|x_{n+1} - f\| &\leq (1-\alpha_n)\|x_n - f\| + \alpha_n\|y_n - f\| \\ &\leq (1-\alpha_n)\|x_n - f\| + \alpha_n\|x_n - f\| \\ &= \|x_n - f\|.\end{aligned}$$

Thus, for $\alpha_n \neq 0, \{\|\alpha_n - f\|\}$ is a nonincreasing sequence. Then, $\lim_{n\to\infty}\|x_n - f\|$ exists.

Now we show that $\{x_n\}$ converges weakly to a common fixed point of T and I. The sequence $\{x_n\}$ contains a subsequence which converges weakly to a point in K. Let $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be to subsequences of $\{x_n\}$ which converge weakly to f and q, respectively. We will show that $f = q$. Suppose that X satisfies Opial's condition and that $f \neq q$ is in weak limit set of the sequence $\{x_n\}$. Then $\{x_{n_k}\} \to f$ and $\{x_{m_k}\} \to q$, respectively. Since $\lim_{n\to\infty}\|x_n - f\|$ exists for any $f \in F(T)\cap F(I)$, by Opial's condition, we conclude that

$$\begin{aligned}\lim_{n\to\infty}\|x_n - f\| &= \lim_{k\to\infty}\|x_{n_k} - f\| < \lim_{k\to\infty}\|x_{n_k} - q\| \\ &= \lim_{n\to\infty}\|x_n - q\| = \lim_{j\to\infty}\|x_{m_j} - q\| \\ &< \lim_{j\to\infty}\|x_{m_j} - f\| = \lim_{n\to\infty}\|x_n - f\|.\end{aligned}$$

This is a contradiction. Thus $\{x_n\}$ converges weakly to an element of $F(T)\cap F(I)$. □

Theorem 2.2 *Let K be a closed convex bounded subset of a uniformly convex Banach space X, which satisfies Opial's condition, and let T, I self-mappings of K with T be an I-quasi-nonexpansive mapping, I a nonexpansive on K. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of Ishikawa defined by (1.2) converges weakly to common fixed point of $F(T)\cap F(I)$.*

Proof. Putting $\gamma_n = 0$ in Theorem 2.1, we obtain the desired result. □

If $\gamma_n = 0$ and $\beta_n = 0$ in Theorem 2.1, we obtain the following corollary :

Corollary 2.3 *(cf. Theorem 2.1 [12]) Let K be a closed convex bounded subset of a uniformly convex Banach space X, which satisfies Opial's condition, and let T, I self-mappings of K with T be an I-quasi-nonexpansive mapping, I a nonexpansive on K. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of Mann converges weakly to common fixed point of $F(T)\cap F(I)$.*

Acknowledgement. The authors would like to thanks to **the King Mongkut's University of Technology Thonburi (KMUTT) Research Fund 2008** for financial support.

References

[1] J.-B. Baillon, *Un th´eor`eme de type ergodique pour les contractions non lin´eaires dans un espacede Hilbert*, Comptes Rendus de l'Acad`emie des Sciences de Paris, S´erie A **280** (1975), no. 22, 1511–1514.

[2] W. G. Dotson Jr., *On the Mann iterative process*, Transactions of the American Mathematical Society **149** (1970), no. 1, 65–73.

[3] M. K. Ghosh and L. Debnath, *Convergence of Ishikawa iterates of quasi-nonexpansive mappings*, Journal of Mathematical Analysis and Applications **207** (1997), no. 1, 96–103.

[4] W. A. Kirk, *Remarks on approximation and approximate fixed points in metric fixed point theory*, Annales Universitatis Mariae Curie-Skłodowska. Sectio A **51** (1997), no. 2, 167–178.

[5] H. Kiziltunc and M. Ozdemir, On convergence theorem for nonself I -nonexpansive mapping in Banach spaces, Applied Mathematical Sciences, Vol. 1, 2007, no. 48, 2379 - 2383

[6] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bulletin of the American Mathematical Society **73** (1967), 591–597.

[7] A. Pazy, *On the asymptotic behavior of iterates of nonexpansive mappings in Hilbert space*, Israel Journal of Mathematics **26** (1977), no. 2, 197–204.

[8] W. V. Petryshyn and T. E. Williamson Jr., *Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings*, Journal of Mathematical Analysis and Applications **43** (1973), 459–497.

[9] N. Shahzad, *Generalized I-nonexpansivemaps and best approximations in Banach spaces*, Demonstratio Mathematica **37** (2004), no. 3, 597–600.

[10] S. Temir and O. Gul, *Convergence theorem for I-asymptotically quasi-nonexpansive mapping in Hilbert space*, to appear in Journal of Mathematical Analysis and Applications.

[11] H. Zhou, R. P. Agarwal, Y. J. Cho, and Y. S. Kim, *Nonexpansive mappings and iterative method in uniformly convex Banach spaces*, Georgian Mathematical Journal **9** (2002), no. 3, 591–600.

[12] B. E. Rhoades and Seyit Temir, *Convergence theorems for I-nonexpansive mapping*, International Journal of Mathematical Sciences, Volume 2006, Article ID 63435, Pages 1-4.

Received: March 26, 2008