On Intrinsic Compactness of Regular Solutions Space of a Class of Higher Order Operator-Differential Equations

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Abstract
In the paper we give definition of intrinsic compactness of regular solutions spaces of a class of higher order homogeneous operator-differential equations and find sufficient conditions for intrinsic compactness of these solutions, at which the equation describes the corrosive fracture process of metals in aggressive media and the principal part of the equation is of complicated character. Then we prove a Phragmen-Lindeloff type theorem for regular solutions of a class of higher order operator-differential equations. The found conditions are expressed by the operator coefficients of the equation.

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1. Introduction

Many problems of mechanics and mathematical physics are bound up with investigations of solvability of operator-differential equations. We can indicate the following papers. As is known, stress-strain state of a plate can be divided into internal and boundary layers [1-4]. Construction of a boundary layer is connected with sequential solution of plane problems of elasticity theory in a semi-strip. In Papkovich’s paper [5] and in others, a boundary value problem of elasticity theory in a semi-strip \( x > 0, |y| \leq 1 \) is reduced to determination of Airy biharmonic function that is sought in the form

\[ u = \sum_{\text{Im} \sigma_k > 0} C_k \varphi_k(y) e^{i \sigma_k x}, \]
where \( \varphi_k \) are the Papkovich functions [5-6], \( \sigma_k \) are eigen values of some self-adjoint boundary value problem, \( C_k \) are the unknown constants. In this connection in [6] there is a problem on representation of a pair of functions \( f_1 \) and \( f_2 \) in the form

\[
f_1 = \sum_{k=1}^{\infty} C_k P_k \varphi_k, \quad f_2 = \sum_{k=1}^{\infty} C_k Q_k \varphi_k,
\]

where \( P_k \) and \( Q_k \) are differential operators defined by the boundary conditions for \( x = 0 \). Some sufficient conditions of uniform convergence of the expansion (1) for the cases when the coefficients \( C_k \) are obviously determined by means of the generalized orthogonality, are given in the paper [7]. In [8] the coefficients \( C_k \) are uniquely determined by the boundary values of a biharmonic function and their derivatives. The solution of mechanics and quantum mechanics problems are also bound up with investigation of completeness of some part of eigen and adjoint vectors of operator pencils. The papers of G.V.Radzievskii [9], A.S.Markus [10] and others have been devoted to these questions. In the paper [11] P.D.Lax gives definition of intrinsic compactness for some spaces of solutions at infinite interval and indicates its close relation with Phragmen-Lideloff principle for the solutions of elliptic equations. In our case, the obtained results essentially differ from the conditions given in these papers.

2. Problem statement

Let \( H \) be a separable Hilbert space, \( A \) be a positive – definite self-adjoint operator in \( H \) with domain of definition \( D(\text{A}) \). By \( L_2(R_+;H) \) \((R_+ = (0, +\infty))\) we denote a Hilbert space of vector-functions \( f(t) \) determined in \( R_+ \) almost everywhere with values in \( H \), measurable, square integrable in the Bochner’s sense, moreover

\[
\|f\|_{L_2(R_+;H)} = \left( \int_0^{+\infty} \|f(t)\|^2_H dt \right)^{1/2} < +\infty.
\]

Let’s define a Hilbert space \( W_l^2(R_+;H) \) \((l \geq 1)\) [12]

\[
W_l^2(R_+;H) = \{u \ | \ u^{(l)} \in L_2(R_+;H), \ A^l u \in L_2(R_+;H)\}
\]

with norm

\[
\|u\|_{W_l^2(R_+;H)} = \left( \|u^{(l)}\|^2_{L_2(R_+;H)} + \|A^l u\|^2_{L_2(R_+;H)} \right)^{1/2}.
\]
Further we define the Hilbert space

$$W^j_2(R_+; H; \{\nu\}_{\nu=0}^{2m-1}) = \{ u \mid u \in W^j_2(R_+; H), u^{(\nu)}(0) = 0, \nu = 0, 2m-1 \}.$$ 

The spaces \(L_2((a, b); H)\) and \(W^j_2((a, b); H)\) in the case \(0 \leq a < b < +\infty\) are determined in a similar way.

In a separable Hilbert space \(H\) consider an operator-differential equation

$$P \left( \frac{d}{dt} \right) u(t) \equiv \left( -\frac{d^2}{dt^2} + A^2 \right)^m u(t) + \sum_{j=1}^{2m} A_j u^{(2m-j)}(t) = 0,$$

where \(u(t)\) is a vector-valued function determined in \(R_+\) with values from \(H\), the derivatives \(u^{(j)}(t)\) \((j = 1, 2m)\) are understood in the sense of distributions theory, the operator coefficients \(A_j\) \((j = 1, 2m)\) satisfy the following conditions:

1) \(A\) is a positive-definite self-adjoint operator;
2) \(A^{-1}\) is a completely continuous operator;
3) the operators \(A_j\) \((j = 1, 2m)\) are bounded in \(H\).

The equation (1) describes a corrosive fracture process in aggressive media that was investigated in the paper [13].

In the present we’ll give definition of intrinsic compactness of the space of regular solutions of homogeneous equations (1) and give intrinsic compactness conditions. Notice that properties of interinsic compactness is closely bound up with Phragmen-Lindeloff type theorems. Such theorems for abstract equations were obtained in [14-15]. The main difference of our case from the papers mentioned above is that equation (1) has a complicated-multiple characteristic and the found conditions are expressed by the operator coefficients of the equation.

**Definition 1.** If a vector function \(u \in W^{2m}_2(R_+; H)\) satisfies the equation (1) almost everywhere in \(R_+\), it is said to be a regular solution of the equation (1).

By \(Ker(P, R_+)\) we define a set of regular solutions of the equation (1). Obviously \(Ker(P, R_+)\) is a closed subspace of the space \(W^{2m}_2(R_+; H)\) and invariant with respect to shift, i.e. if \(u(t) \in Ker(P, R_+)\), for any \(\eta > 0\) the vector-function \(u(t + \eta) \in Ker(P, R_+)\). On the other hand, the expressions \(Ker(P, R_+) \subset W^{2m}_2(R_+; H) \subset W^{2m-1}_2(R_+; H)\) hold. Let \(L(P, R_+)\) be a closure of the space \(Ker(P, R_+)\) by the norm of the space \(W^{2m-1}_2(R_+; H)\) weaker than the norm of the space \(W^{2m}_2(R_+; H)\).
Definition 2. Let $0 < a < a' < b' < b < \infty$ and $M > 0$ be any number. If the set $Q_M = \left\{ u \mid u \in L(P, R_+), \| u \|_{W_2^{2m-1}(a,b); H)} \leq M \right\}$ is a compact set in the space $W_2^{2m-1}((a', b'); H)$, we’ll say that $L(P, R_+)$ is an intrinsically compact set.

3. The main result

Now, let’s prove the basic theorem. It holds

Theorem. Let the conditions 1)-3) be satisfied, and for the vector-functions $u(t) \in W_2^{2m} (R_+; H; \{ \nu \}_{\nu=0}^{2m})$ the inequality

$$\| P \left( \frac{d}{dt} \right) u(t) \|_{L_2(R_+; H)} \geq \text{const} \| u \|_{W_2^{2m}(R_+; H)},$$

(3)

hold. Then, the space $L(P, R_+)$ is intrinsically compact and there exists such a number $\omega_0 > 0$ that for any $u(t) \in L(P, R_+)$ the estimation

$$\int_0^\infty e^{2\omega_0 t} \left( \| u^{(2m-1)}(t) \|_H^2 + \| A^{2m-1}u(t) \|_H^2 \right) dt < \infty$$

holds.

Proof. Let $u(t) \in L(P, R_+)$. Then obviously, for $\eta > 0$, $u(t+\eta) \in \text{Ker}(P, R_+)$, i.e. $\text{Ker}(P, R_+)$ is invariant with respect to shift. Define the scalar function $\varphi(t) \in C^\infty (a; b)$. It holds the inequality

$$\varphi(t) = \begin{cases} 1, & t \in (a', b'), \\ 0, & t \in (a, b) \setminus (a', b'). \end{cases}$$

Then for $u(t) \in L(P, R_+)$ the vector-function $v(t) = \varphi(t) u(t) \in W_2^{2m} (R_+; H; \{ \nu \}_{\nu=0}^{2m})$. Consequently, from inequality (2) we get

$$\| P \left( \frac{d}{dt} \right) v(t) \|_{L_2(R_+; H)} \geq \text{const} \| v \|_{W_2^{2m}(R_+; H)}.$$

Since $v(t) = 0$ for $t \in R_+ \setminus (a, b)$, we can write this inequality in the form

$$\| P \left( \frac{d}{dt} \right) v(t) \|_{L_2(R_+; H)} = \| P \left( \frac{d}{dt} \right) (\varphi u) \|_{L_2(R_+; H)} \geq \text{const} \| \varphi u \|_{W_2^{2m}(R_+; H)} =$$

$$= \text{const} \| \varphi u \|_{W_2^{2m}((a, b); H)} = \text{const} \left( \int_a^b \left( \| (\varphi u)^{(2m)} \|_H^2 + \| A^{2m}(\varphi u) \|_H^2 \right) dt \right)^{1/2}$$

$$\geq \text{const} \left( \int_{a'}^{b'} \left( \| (\varphi u)^{(2m)} \|_H^2 + \| A^{2m}(\varphi u) \|_H^2 \right) dt \right)^{1/2} =$$
bounded in the space $W^{2m}((a',b');H)$. Therefore, $Q$ is compact in $\mathbb{K}$.

On the other hand, it is easy to see

$$P(d/dt)\, v(t) = P(d/dt)\, (\varphi u)(t) = \varphi(t)P(d/dt)\, u(t) + \sum_{s=1}^{j} \left( \sum_{j=1}^{2m} q_{s,j}\varphi^{(s)}(t)\tilde{A}_j u^{(j-s)}(t) \right),$$

where $q_{s,j}$ are definite numbers, and $\tilde{A}_j = \left\{ A_j + C_m^j/2 (-1)^j/t^j, A_j, j = 2k - 1, k = 1, m. \right.$

Since $P(d/dt)\, u(t) = 0$, then

$$\|P(d/dt)\, v(t)\|_{L^2(R^+;H)} = \left\| \sum_{s=1}^{j} \left( \sum_{j=1}^{2m} q_{s,j}\varphi^{(s)}(t)\tilde{A}_j u^{(j-s)}(t) \right) \right\|_{L^2(R^+;H)} = \left\| \sum_{s=1}^{j} \left( \sum_{j=1}^{2m} q_{s,j}\varphi^{(s)}(t)\tilde{A}_j u^{(j-s)}(t) \right) \right\|_{L^2((a,b);H)}.$$

Since $\max_t |\varphi^{(s)}(t)| < const, (1 \leq s < 2m)$, and by the theorem on intermediate derivatives [12]

$$\left\| \tilde{A}_j u^{(j-s)}(t) \right\|_{L^2((a,b);H)} = \left\| \tilde{A}_j \cdot A^{-j} \cdot A^j d/dt u^{(j-s)}(t) \right\|_{L^2(R^+;H)} \leq \left\| \tilde{A}_j \cdot A^{-j} \right\|_{L^2((a,b);H)} \leq const \left\| u \right\|_{W^{2m-1}((a,b);H)}.$$

Thus

$$\left\| u \right\|_{W^{2m-1}((a,b);H)} \geq const \left\| u \right\|_{W^{2m}((a',b');H)}.$$

We use inequality (4) and prove the intrinsic compactness of $\text{Ker}(P,R_+)$. Denote $Q'_m = Q_m \cap \text{Ker}(P,R_+)$ and show that $Q'_m$ is compact with respect to the norm $\| u \|_{W^{2m-1}((a',b');H)}$. It follows from inequality (4) the set $Q'_m$ is bounded in the space $W^{2m}_2((a',b');H)$, since

$$\| u \|_{W^{2m}_2((a',b');H)} \leq const \left\| u \right\|_{W^{2m-1}((a,b);H)} \leq Mconst \left\| u \right\|_{W^{2m-1}((a,b);H)}.$$

The operator $A^{-1}$ is completely continuous, the embedding

$$W^{2m}_2((a',b');H) \subset W^{2m-1}_2((a',b');H)$$

is compact in [14]. Therefore, $Q'_m$ is a compact set in $W^{2m-1}_2((a',b');H)$, i.e. $\text{Ker}(P,R_+)$ is intrinsically compact.

The theorem is proved.
References


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