On Subordination Theorems for New Classes of Normalize Analytic Functions

Rabha W. Ibrahim and Maslina Darus

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600, Selangor Darul Ehsan, Malaysia
rabhaibrahim@yahoo.com, maslina@ukm.my

Abstract

New classes of analytic functions are defined. We give some applications of first order differential subordination obtain sufficient conditions for normalized analytic functions.

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1 Introduction and preliminaries.

Let $A_\alpha^+$ be the class of all normalized analytic functions $F(z)$ in the open disk $U := \{z \in \mathbb{C}, |z| < 1\}$, take the form

$$F(z) = z + \sum_{n=2}^{\infty} a_n z^{n+\alpha-1}, \quad 0 < \alpha \leq 1,$$

satisfying $F(0) = 0$ and $F'(0) = 1$. And let $A_\alpha^-$ be the class of all normalized analytic functions $F(z)$ in the open disk $U$ take the form

$$F(z) = z - \sum_{n=2}^{\infty} a_n z^{n+\alpha-1}, \quad a_n \geq 0; \quad n = 2, 3, ...,$$

satisfying $F(0) = 0$ and $F'(0) = 1$. With a view to recalling the principle of subordination between analytic functions, let the functions $f$ and $g$ be analytic
in $U$. Then we say that the function $f$ is *subordinate* to $g$ if there exists a
Schwarz function $w(z)$, analytic in $U$ such that

$$f(z) = g(w(z)), \ z \in U.$$  

We denote this subordination by

$$f \prec g \ or \ f(z) \prec g(z), \ z \in U.$$  

If the function $g$ is univalent in $U$ the above subordination is equivalent to

$$f(0) = g(0) \ and \ f(U) \subset g(U).$$

Let $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and let $h$ be univalent in $U$. Assume that $p, \phi$ are analytic
and univalent in $U$ if $p$ satisfies the differential superordination

$$h(z) \prec \phi(p(z)), zp'(z), z^2p''(z); z), \quad (1.1)$$

then $p$ is called a solution of the differential superordination.(If $f$ is subordinate
to $g$, then $g$ is called to be superordinate to $f$.) An analytic function $q$ is called a *subordinant* if $q \prec p$ for all $p$ satisfying (1). An univalent function $q$ such that $p \prec q$ for all subordinants $p$ of (1) is said to be the best subordinant.

Let $\mathcal{A}$ be the class of analytic functions of the form $f(z) = z + a_2 z^2 + \ldots$. Ali
et al [1] have used the results of Bulboac˘a [2] and obtain sufficient conditions
for certain normalized analytic functions $f(z) \in \mathcal{A}$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where $q_1$ and $q_2$ are given univalent functions in $U$ with $q_1(0) = q_2(0) = 1$. The
main object of the present work is to apply a method based on the differential
subordination in order to derive sufficient conditions for functions $F \in \mathcal{A}_\alpha^+$ and $F \in \mathcal{A}_\alpha^-$ to satisfy

$$\frac{zF'(z)}{F(z)} \prec q(z) \quad (1.2)$$

where $q(z)$ is given univalent function in $U$ with $q(z) \neq 0$. Moreover, we give
applications for these results in fractional calculus. We shall need the following
known results.

**Lemma 1.1.** [3] Let $q(z)$ be univalent in the unit disk $U$ and $\theta$ and $\phi$ be
analytic in a domain $D$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) := zq'(z)\phi(q(z)), h(z) := \theta(q(z)) + Q(z)$. Suppose that

1. $Q(z)$ is starlike univalent in $U$, and
2. \( \Re \frac{zh'(z)}{q(z)} > 0 \) for \( z \in U \).
If \( \theta(p(z)) + zp'(z) \phi(p(z)) \prec \theta(q(z)) + zq'(z) \phi(q(z)) \) then \( p(z) \prec q(z) \) and \( q(z) \) is the best dominant.

**Lemma 1.2.** [4] Let \( q(z) \) be convex univalent in the unit disk \( U \) and \( \psi \) and \( \gamma \in \mathbb{C} \) with \( \Re \{1 + \frac{q''(z)}{q'(z)} + \frac{\psi}{\gamma}\} > 0 \). If \( p(z) \) is analytic in \( U \) and \( \psi p(z) + \gamma z p'(z) \prec \psi q(z) + \gamma z q'(z) \), then \( p(z) \prec q(z) \) and \( q \) is the best dominant.

### 2 Subordination results.

In this section, we study the subordination between analytic functions.

**Theorem 2.1.** Let the function \( q(z) \) be univalent in the unit disk \( U \) such that \( q(z) \neq 0 \), and

\[
\Re \{1 + \frac{q''(z)}{q'(z)} - \frac{q'(z)}{q(z)} + \frac{b}{\gamma} q(z) + \frac{2c}{\gamma} q^2(z) + \frac{3d}{\gamma} q^3(z)\} > 0, \quad b, c, d \in \mathbb{C}, \quad \gamma \neq 0.
\]  

(2.3)

Suppose that \( \frac{q'(z)}{q(z)} \) is starlike univalent in \( U \). If \( F \in \mathcal{A}_\alpha^+ \) satisfies the subordination

\[
a + b \frac{z F'(z)}{F(z)} + c \left[ \frac{z F'(z)}{F(z)} \right]^2 + d \left[ \frac{z F'(z)}{F(z)} \right]^3 + \gamma \left[1 + \frac{z F''(z)}{F'(z)} - \frac{z F'(z)}{F(z)} \right] \prec a + bq(z) + cq^2(z) + dq^3(z) + \gamma \frac{z q'(z)}{q(z)}.
\]

Then

\[
\frac{z F'(z)}{F(z)} \prec q(z), \quad z \in U, \quad F(z) \neq 0
\]

and \( q(z) \) is the best dominant.

**Proof.** Let the function \( p(z) \) be defined by

\[
p(z) := \frac{z F'(z)}{F(z)}, \quad F(z) \neq 0, \quad z \in U.
\]

By setting

\[
\theta(\omega) := a + b\omega + c\omega^2 + d\omega^3 \quad \text{and} \quad \phi(\omega) := \frac{\gamma}{\omega}, \quad a \neq 0,
\]
it can easily be observed that $\theta(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C} - \{0\}$ and that $\phi(\omega) \neq 0$, $\omega \in \mathbb{C} - \{0\}$. Also we obtain

$$Q(z) = zq'(z)\phi(q(z)) = \gamma \frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = a + bq(z) + cq^2(z) + dq^3(z) + \gamma \frac{zq'(z)}{q(z)}.$$ 

It is clear that $Q(z)$ is starlike univalent in $U$,

$$\Re\left\{\frac{zh'(z)}{Q(z)} = \Re\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{b}{\gamma}q(z) + \frac{2c}{\gamma}q^2(z) + \frac{3d}{\gamma}q^3(z)\} > 0.\right.$$

By a straightforward computation, we have

$$a + bp(z) + cp^2(z) + dp^3(z) + \gamma \frac{zp'(z)}{p(z)} = a + b\frac{ZF'(z)}{F(z)} + c[\frac{zF'(z)}{F(z)}]^2 + d[\frac{zF'(z)}{F(z)}]^3$$

$$+ \gamma[1 + \frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)}]$$

$$< a + bq(z) + cq^2(z) + dq^3(z) + \gamma \frac{zq'(z)}{q(z)}.$$ 

Then by the assumption of the theorem we have that the assertion of the theorem follows by an application of Lemma 1.1.

**Corollary 2.1.** Assume that (3) holds and $q$ is convex univalent in $U$. If $F \in A^+_a$ and

$$a + b\frac{ZF'(z)}{F(z)} + c[\frac{zF'(z)}{F(z)}]^2 + d[\frac{zF'(z)}{F(z)}]^3 + \gamma[1 + \frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)}]$$

$$< a + b\frac{1 + Az}{1 + Bz} + c[\frac{1 + Az}{1 + Bz}]^2 + d[\frac{1 + Az}{1 + Bz}]^3 + \gamma \frac{z(A - B)}{(1 + Az)(1 + Bz)}$$

then

$$\frac{zF'(z)}{F(z)} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1$$

and $q(z) = \frac{1 + Az}{1 + Bz}$ is the best dominant.
Corollary 2.2. Assume that (3) holds and \( q \) is convex univalent in \( U \). If \( F \in \mathcal{A}_+^\alpha \) and
\[
a + b \frac{zF'(z)}{F(z)} + c \left[ \frac{zF'(z)}{F(z)} \right]^2 + d \left[ \frac{zF'(z)}{F(z)} \right]^3 + \gamma \left[ 1 + \frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)} \right] \prec a + b \left[ \frac{1 + z}{1 - z} \right]^\mu + c \left[ \frac{1 + z}{1 - z} \right]^{2\mu} + d \frac{1 + z}{1 - z}^{3\mu} + \frac{2\mu\gamma z}{(1 + z)^2}
\]
for \( z \in U, \mu \neq 0 \), then
\[
\frac{zF'(z)}{F(z)} \prec \left[ \frac{1 + z}{1 - z} \right]^\mu
\]
and \( q(z) = [1 + \frac{z}{1 - z}]^\mu \) is the best dominant.

Corollary 2.3. Assume that (3) holds and \( q \) is convex univalent in \( U \). If \( F \in \mathcal{A}_+^\alpha \) and
\[
a + b \frac{zF'(z)}{F(z)} + c \left[ \frac{zF'(z)}{F(z)} \right]^2 + d \left[ \frac{zF'(z)}{F(z)} \right]^3 + \gamma \left[ 1 + \frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)} \right] \prec a + b e^{\mu \text{Az}} + c e^{2\mu \text{Az}} + d e^{3\mu \text{Az}} + \mu \gamma \text{Az}
\]
for \( z \in U, \mu \neq 0 \), then
\[
\frac{zF'(z)}{F(z)} \prec e^{\mu \text{Az}}
\]
and \( q(z) = e^{\mu \text{Az}} \) is the best dominant.

Theorem 2.2. Let the function \( q(z) \) be convex univalent in the unit disk \( U \) such that
\[
\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\gamma} \right\} > 0, \quad \gamma \neq 0.
\]
(2.4)
Suppose that \( \frac{zF'(z)}{F(z)} \) is analytic in \( U \). If \( F \in \mathcal{A}_-^\alpha \) satisfies the subordination
\[
\frac{zF'(z)}{F(z)} + \gamma \frac{zF'(z)}{F(z)} \left[ 1 + \frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)} \right] \prec q(z) + \gamma zq'(z).
\]
Then
\[
\frac{zF'(z)}{F(z)} \prec q(z), \quad z \in U, \; F(z) \neq 0
\]
and $q(z)$ is the best dominant.

**Proof.** Let the function $p(z)$ be defined by

$$p(z) := \frac{zF'(z)}{F(z)}, \quad F(z) \neq 0, \quad z \in U.$$  

By setting $\psi = 1$, it can easily observed that

$$p(z) + \gamma z p'(z) = \frac{zF'(z)}{F(z)} + \gamma \frac{zF'(z)}{F(z)} \left[ 1 + \frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)} \right] \prec q(z) + \gamma z q'(z).$$

Then by the assumption of the theorem we have that the assertion of the theorem follows by an application of Lemma 1.2.

**Corollary 2.4.** Assume that (4) holds and $q$ is convex univalent in $U$. If $F \in A_\alpha$ and

$$\frac{zF'(z)}{F(z)} + \gamma \frac{zF'(z)}{F(z)} \left[ 1 + \frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)} \right] \prec \frac{1 + Az}{1 + Bz} + \gamma z(A - B) \left( \frac{1 + Bz}{1 + Bz} \right)^2$$

then

$$\frac{zF'(z)}{F(z)} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1$$

and $q(z) = \frac{1 + Az}{1 + Bz}$ is the best dominant.

**Corollary 2.5.** Assume that (4) holds and $q$ is convex univalent in $U$. If $F \in A_\alpha$ and

$$\frac{zF'(z)}{F(z)} + \gamma \frac{zF'(z)}{F(z)} \left[ 1 + \frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)} \right] \prec \left( 1 + z \right)^\mu + 2\gamma z \frac{(1 + z)^{\mu - 1}}{(1 - z)^{\mu + 1}}$$

for $z \in U, \mu \neq 0$, then

$$\frac{zF'(z)}{F(z)} \prec \left( 1 + z \right)^\mu \frac{1 + z}{1 - z}$$

and $q(z) = \frac{1 + z}{1 - z}$ is the best dominant.

**Corollary 2.6.** Assume that (4) holds and $q$ is convex univalent in $U$. If $F \in A_\alpha$ and

$$\frac{zF'(z)}{F(z)} + \gamma \frac{zF'(z)}{F(z)} \left[ 1 + \frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)} \right] \prec e^{\mu Az} + \mu A ze^{\mu Az}$$

for $z \in U, \mu \neq 0$, then

$$\frac{zF'(z)}{F(z)} \prec e^{\mu Az}$$

and $q(z) = e^{\mu Az}$ is the best dominant.
3 Applications.

In this section, we introduce some applications of section (2) containing fractional integral operators. Assume that $f(z) = \sum_{n=2}^{\infty} \varphi_n z^n$ and let us begin with the following definitions

**Definition 3.1.** [5] The fractional integral of order $\alpha$ is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad 0 \leq \alpha < 1,$$

where the function $f(z)$ is analytic in simply-connected region of the complex $z$-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$. Note that, $I_z^\alpha f(z) = f(z) \times \frac{z^{\alpha-1}}{\Gamma(\alpha)}$, for $z > 0$ and $0$ for $z \leq 0$ (see [6]).

From Definition 3.1, we have

$$I_z^\alpha f(z) = f(z) \times \frac{z^{\alpha-1}}{\Gamma(\alpha)} = \sum_{n=2}^{\infty} a_n z^{n+\alpha-1}$$

where $a_n := \frac{\varphi_n}{\Gamma(\alpha)}$, for all $n = 2, 3, \ldots$, thus $z + I_z^\alpha f(z) \in A^+$ and $z - I_z^\alpha f(z) \in A^-$ ($\varphi_n \geq 0$), then we have the following results

**Theorem 3.1.** Let the assumptions of Theorem 2.1 hold, then

$$\frac{z[1 + I_z^\alpha f'(z)]}{[z + I_z^\alpha f(z)]} \prec q(z),$$

and $q(z)$, is the best dominant.

**Proof.** Let the function $F(z)$ be defined by

$$F(z) := z + I_z^\alpha f(z), \quad z \in U, \ F(z) \neq 0.$$ 

Since $f(0) = 0$, one can verify that $[I_z^\alpha f(z)]' = I_z^\alpha f'(z)$ then we obtain the result.

**Theorem 3.2.** Let the assumptions of Theorem 2.2 hold, then

$$\frac{z[1 - I_z^\alpha f'(z)]}{[z - I_z^\alpha f(z)]} \prec q(z),$$

and $q(z)$, is the best dominant.

**Proof.** Let the function $F(z)$ be defined by

$$F(z) := z - I_z^\alpha f(z), \quad z \in U, \ F(z) \neq 0.$$
Let $F(a, b; c; z)$ be the Gauss hypergeometric function (see [7]) defined, for $z \in U$, by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n,$$

where is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a + n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a + 1)(a + 2)\ldots(a + n - 1), & (n \in \mathbb{N}). \end{cases}$$

We need the following definitions of fractional operators in the Saigo type fractional calculus (see [8],[9]).

**Definition 3.2.** For $\alpha > 0$ and $\beta, \eta \in \mathbb{R}$, the fractional integral operator $I_{0, z}^{\alpha, \beta, \eta}$ is defined by

$$I_{0, z}^{\alpha, \beta, \eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z - \zeta)^{\alpha-1} F(\alpha + \beta, -\eta; \alpha; 1 - \frac{\zeta}{z}) f(\zeta) d\zeta$$

where the function $f(z)$ is analytic in a simply-connected region of the $z$–plane containing the origin, with the order

$$f(z) = O(|z|^\epsilon)(z \to 0), \quad \epsilon > \max\{0, \beta - \eta\} - 1$$

and the multiplicity of $(z - \zeta)^{\alpha-1}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

From Definition 3.2, with $\beta < 0$, we have

$$I_{0, z}^{\alpha, \beta, \eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z - \zeta)^{\alpha-1} F(\alpha + \beta, -\eta; \alpha; 1 - \frac{\zeta}{z}) f(\zeta) d\zeta$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_n (-\eta)_n}{(\alpha)_n (1)_n} \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z - \zeta)^{\alpha-1} (1 - \frac{\zeta}{z})^n f(\zeta) d\zeta$$

$$:= \sum_{n=0}^{\infty} \frac{B_n z^{-\alpha-\beta-n}}{\Gamma(\alpha)} \int_0^z (z - \zeta)^{\alpha+n-1} f(\zeta) d\zeta$$

$$= \sum_{n=0}^{\infty} \frac{B_n z^{-\beta-1}}{\Gamma(\alpha)} f(\zeta)$$

$$:= \frac{\mathcal{B}}{\Gamma(\alpha)} \sum_{n=2}^{\infty} \varphi_n z^{n-\beta-1}$$

where $\mathcal{B} := \sum_{n=0}^{\infty} B_n$. Denote $a_n := \frac{B_n \varphi_n}{\Gamma(\alpha)}$, $\forall n = 2, 3, \ldots$, and let $\alpha = -\beta$ thus $z + I_{0, z}^{\alpha, \beta, \eta} f(z) \in \mathcal{A}_\alpha^+$ and $z - I_{0, z}^{\alpha, \beta, \eta} f(z) \in \mathcal{A}_\alpha^-$ ($\varphi_n \geq 0$), then we have the following results.
Theorem 3.3. Let the assumptions of Theorem 2.1 hold, then
\[
\frac{z[z + I^\alpha_{0,z} f(z)]'}{[z + I^\alpha_{0,z} f(z)]} < q(z),
\]
and \(q(z)\) is the best dominant.

**Proof.** Let the function \(F(z)\) be defined by
\[
F(z) := z + I^\alpha_{0,z} f(z), \quad z \in U, \quad F(z) \neq 0.
\]

Theorem 3.4. Let the assumptions of Theorem 2.2 hold, then
\[
\frac{z[z - I^\alpha_{0,z} f(z)]'}{[z - I^\alpha_{0,z} f(z)]} < q(z),
\]
and \(q(z)\) is the best dominant.

**Proof.** Let the function \(F(z)\) be defined by
\[
F(z) := z - I^\alpha_{0,z} f(z), \quad z \in U, \quad F(z) \neq 0.
\]

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**References**


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