Generalized RTS with Discretionary and Nondiscretionary Inputs and Outputs

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Abstract

In some situations, some inputs and outputs are nondiscretionary. Therefore, following Zarepisheh and Soleimani-damaneh [Euro. J. Opera. Res. (2007)], which uses the linear programming problem for determining RTS with discretionary data, in this paper we introduce a procedure for determining RTS with discretionary and nondiscretionary inputs and outputs.

Keywords: Linear programming; Data envelopment analysis; Parametric analysis; Returns to scale; Efficiency; Nondiscretionary data

1 Introduction

Returns to scale (RTS) is an important topic in performance analysis, which helps managers to make decisions about the expansion or contraction of the operation of the Decision Making Unit (DMU) under assessment. RTS can provide useful information on the optimal size of DMUs, or on whether small in size DMUs over- or under-perform larger ones, and vice versa, i.e., it is used to determine whether a technically efficient DMU can improve its productivity by resizing the scale of its operations. In economics, RTS are sometimes defined using the notion of elasticity (Starrett [3]). If elasticity is greater than one, then increasing returns to scale prevail for that particular DMU; if it is equal

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to one, then constant returns to scale prevail for that particular DMU; and if it is less than one, then decreasing returns to scale prevail for that particular DMU. In fact, the elasticity measure exhibits the rate of proportional variation of outputs with respect to the proportional variation of inputs in a local sense; i.e., in a sufficiently small interval of variations Hadjicostas and Soteriou [2].

Zarepisheh and Soleimani-damaneh [4] propose a procedure for generalized RTS of DMUs with discretionary data, in this paper we introduce a method for determining generalized RTS with discretionary and nondiscretionary inputs and outputs. The rest of this paper organized as follows: in Section 2 we propose a method for determining RTS of DMUS with nondiscretionary inputs and outputs by BCC model. Sections 3 contain a numerical example. Finally Section 3 includes some conclusions.

2 Estimation of the rate of variation

In some situations, some inputs and outputs are nondiscretionary. These data are beyond of the control of a DMU’s management, which also need to be considered for efficiency evaluation. The BCC model[1] in input oriented with nondiscretionary is as follows:

\[
\beta^*_o = \min \beta_o \\
\text{s.t. } \sum_{j=1}^n \lambda_j x_{ij} \leq \beta_o x_{io}^D, \quad i \in D \\
\sum_{j=1}^n \lambda_j x_{ij} \leq \beta_o x_{io}^{ND}, \quad i \in ND \\
\sum_{j=1}^n \lambda_j y_{r} \geq y_{ro}, \quad r = 1, \ldots, s, \\
\sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, \quad j = 1, \ldots, n.
\]

where, \( D \) and \( ND \) represent the associated sets containing discretionary and nondiscretionary inputs, respectively.

Assume that Increasing Returns to Scale (IRS) prevail at DMU\(_o\)(\(x_{io}^D, x_{io}^{ND}, y_o\)). The Right Hand Side (RHS) vector of model (1) is \((0, x_{io}^{ND}, y_o, 1)\). Now in the optimal simplex tableau of (1) we perturb the RHS vector in the direction of \((0, 0, y_o, 0)\), and this leads to \((0, x_{io}^{ND}, y_o, 1+\delta(0, 0, y_o, 0) = (0, x_{io}^{ND}, (1+\delta)y_o, 1)\) \((\delta \geq 0)\) as the new RHS vector. Now using the parametric analysis, we can obtain an interval, \( \delta \in [0, \delta^1] \), such that the optimal value of the perturbed problem is a linear function with respect to \( \delta \in [0, \delta^1] \). Regarding model (1) the optimal value exhibits the minimum proportional increase in the (D) inputs of DMU\(_o\) when all outputs of this DMU are multiplied by \((1 + \delta)\)(for \( \delta \in [0, \delta^1] \)). In fact the optimal value of the perturbed problem is \( \beta(1 + \delta) \) for \( \delta \in [0, \delta^1] \).
3 Numerical example

Consider 6 DMUs with 2 inputs and 1 output as Table 1, where 2\textsuperscript{nd} input is nondiscretionary.

<table>
<thead>
<tr>
<th>DMUs</th>
<th>DMU\textsubscript{A}</th>
<th>DMU\textsubscript{B}</th>
<th>DMU\textsubscript{C}</th>
<th>DMU\textsubscript{D}</th>
<th>DMU\textsubscript{E}</th>
<th>DMU\textsubscript{F}</th>
</tr>
</thead>
<tbody>
<tr>
<td>input\textsubscript{1}</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>9</td>
<td>13</td>
<td>7</td>
</tr>
<tr>
<td>input\textsubscript{2}</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>output</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>2</td>
</tr>
</tbody>
</table>

We consider DMU\textsubscript{A} as unit under assessment. Model (1) corresponding to this DMU is as follows:

\[
\beta^*_A = \min \beta \quad \text{s.t.} \quad \begin{align*}
3\lambda_1 + 4\lambda_2 + 7\lambda_3 + 9\lambda_4 + 13\lambda_5 + 7\lambda_6 & \leq 3\beta, \\
2\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4 + 3\lambda_5 + 3\lambda_6 & \leq 2, \\
\lambda_1 + 3\lambda_2 + 7\lambda_3 + 8\lambda_4 + 9\lambda_5 + 2\lambda_6 & \geq 1, \\
\sum_{j=1}^6 \lambda_j & = 1, \lambda_j \geq 0, \quad j = 1, \ldots, 6.
\end{align*}
\]  

(2)

An optimal tableau for this problem is as,

\[
\begin{array}{cccccccccc}
z & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \beta & s_1 & s_2 & R_3 & R_4 & \text{RHS} \\
\hline
z & 1 & 0 & 0 & -0.33 & -0.83 & -2 & -1.16 & 0 & -0.33 & 0 & 0.16 & 0.83 & 1 \\
\lambda_2 & 0 & 0 & 1 & 3 & 3.5 & 4 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 \\
s_2 & 0 & 0 & 0 & 3 & 2.5 & 5 & 1.5 & 0 & 0 & 1 & 0.5 & -2.5 & 0 \\
\beta & 0 & 0 & 0 & -0.33 & -0.83 & -2 & -1.16 & 1 & -0.33 & 0 & 0.16 & 0.83 & 1 \\
\lambda_1 & 1 & 0 & 0 & -2 & -2.5 & -3 & -0.5 & 0 & 0 & 0 & -0.5 & 1.5 & 1 \\
\end{array}
\]

where \(s_1\) and \(s_2\) are the slack variables of the first and the second constraints, respectively. Also \(R_3\) and \(R_4\) are the artificial variables of the third and fourth constraints. Now we do the parametric analysis, when the RHS vector of the problem is perturbed of \(b' = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \bar{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\), \(\bar{b}' = B^{-1}b' = \begin{pmatrix} 0 & 0 & 0.5 & -0.5 \\ 0 & 1 & 0.5 & -2.5 \\ -0.33 & 0 & 0.16 & 0.83 \end{pmatrix}, \bar{b}' = B^{-1}b' = \begin{pmatrix} 0 \\ 0 & 0 & -0.5 & 1.5 \end{pmatrix}\). Since \(\bar{b}'_1 < 0\) and \(\bar{b}'_1 = 0\), then using the algorithm of the parametric analysis in linear programming, \(\lambda_2\) leaves the basis and since \(\forall j, y_{1j} > 0\) then \(\delta^-_1 = 0\). (perturbed problem is infeasible for any \(\delta > 0\)). Hence there isn’t decreasing in output, we now do the parametric analysis, when the RHS vector of the problem is perturbed in the direction of

\[
\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \bar{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \bar{b}' = B^{-1}b' = \begin{pmatrix} 0 & 0 & 0.5 & -0.5 \\ 0 & 1 & 0.5 & -2.5 \\ -0.33 & 0 & 0.16 & 0.83 \end{pmatrix}, \bar{b}' = B^{-1}b' = \begin{pmatrix} 0 \\ 0 & 0 & -0.5 & 1.5 \end{pmatrix}\).
\[ \delta_1^+ = \frac{1}{-(-0.5)} = 2, \beta(\delta) = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.16 \\ -0.5 \end{pmatrix} \] 

\[ \begin{pmatrix} c_B \bar{b} + \delta c_B \bar{b}' = 1+\delta(0,0,1,0) \end{pmatrix} = 1+0.16\delta \implies m_1^+ = 0.16. \]

Since \( \frac{1}{m_1^+} = 6.25 > 1 \), IRS prevail at DMU_A. The optimal table for \( \delta = \delta_1^+ \) is as follows:

<table>
<thead>
<tr>
<th>z</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_3 )</th>
<th>( \lambda_4 )</th>
<th>( \lambda_5 )</th>
<th>( \lambda_6 )</th>
<th>( \beta )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( R_3 )</th>
<th>( R_4 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-0.33</td>
<td>-0.83</td>
<td>-2</td>
<td>-1.16</td>
<td>0</td>
<td>-0.33</td>
<td>0</td>
<td>0.16</td>
<td>0.83</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>3.5</td>
<td>4</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>-0.5</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>2.5</td>
<td>5</td>
<td>1.5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.5</td>
<td>-2.5</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-0.33</td>
<td>-0.83</td>
<td>-2</td>
<td>-1.16</td>
<td>1</td>
<td>-0.33</td>
<td>0</td>
<td>0.16</td>
<td>0.83</td>
</tr>
<tr>
<td>( \lambda_5 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>-2.5</td>
<td>-3</td>
<td>-0.5</td>
<td>0</td>
<td>0</td>
<td>-0.5</td>
<td>1.5</td>
<td>0</td>
</tr>
</tbody>
</table>

The above table is optimal for \( (x_A^D, x_A^{ND}, (1 + \delta_1^+)y_A) = (3, 2, 3) \).

Regarding the algorithm cost row and the \( \beta \)-row (except for the \( \beta \)-column) are multiplied by \( \frac{1}{\beta(\delta_1^+)} = \frac{1}{1.32} \). The following tableau is obtained:

<table>
<thead>
<tr>
<th>z</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_3 )</th>
<th>( \lambda_4 )</th>
<th>( \lambda_5 )</th>
<th>( \lambda_6 )</th>
<th>( \beta )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( R_3 )</th>
<th>( R_4 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>1</td>
<td>-0.12</td>
<td>0</td>
<td>0</td>
<td>-0.3</td>
<td>-1.13</td>
<td>-0.8</td>
<td>0</td>
<td>-0.25</td>
<td>0.18</td>
<td>0.43</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>0</td>
<td>-3</td>
<td>1</td>
<td>0</td>
<td>-0.25</td>
<td>-0.5</td>
<td>-0.25</td>
<td>0</td>
<td>0</td>
<td>-0.25</td>
<td>1.75</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>0</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>-1.25</td>
<td>0.5</td>
<td>0.75</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-0.25</td>
<td>-0.25</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0</td>
<td>-0.12</td>
<td>0</td>
<td>0</td>
<td>-0.3</td>
<td>-1.13</td>
<td>-0.8</td>
<td>1</td>
<td>-0.25</td>
<td>0.18</td>
<td>0.43</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_5 )</td>
<td>0</td>
<td>-0.5</td>
<td>0</td>
<td>1</td>
<td>1.25</td>
<td>1.5</td>
<td>0.25</td>
<td>0</td>
<td>0</td>
<td>0.25</td>
<td>-0.75</td>
<td>0</td>
</tr>
</tbody>
</table>
Now, we do the parametric analysis in the direction of \( m \). Therefore, \( \beta \)

\[
\delta_2^+ = \frac{1}{-(-0.75)} = 1.33, \beta(\delta) = c_B \bar{b} + \delta c_B \bar{y} = 1 + \delta (0, 0, 1, 0) \left( \begin{array}{c} -0.75 \\ -0.75 \\ 0.54 \\ 0.75 \end{array} \right) = 1 + 0.54 \delta.
\]

Therefor, \( m_2^+ = 0.54 \) and \( \frac{1}{m_2} = 1.85 \). The optimal table for \( \delta = \delta_2^+ \) is as follows:

\[
\begin{array}{cccccccccc}
& z & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_6 & \beta & s_1 & s_2 & R_3 & R_4 & \text{RHS} \\
\hline
z & 1 & -0.12 & 0 & 0 & -0.3 & -1.13 & -0.8 & 0 & -0.25 & 0 & 0.18 & 0.43 & 1.72 \\
\lambda_2 & 0 & -1.5 & 1 & 0 & -0.25 & -0.25 & -0.5 & 0 & 0 & 0 & 0 & 0 & 1.75 \\
s_2 & 0 & -1.5 & 0 & 0 & -1.25 & 0.5 & 0.75 & 0 & 0 & 1 & -25 & -25 & 0 \\
\beta & 0 & -0.12 & 0 & 0 & -0.3 & -1.13 & -0.8 & 1 & -0.25 & 0 & 0.18 & 0.43 & 1.72 \\
\lambda_3 & 0 & -0.5 & 0 & 1 & 1.25 & 1.5 & 0.25 & 0 & 0 & 0 & 0 & 25 & -0.75 \\
\end{array}
\]

\((\beta(\delta_1^+)x_A^P, x_A^{ND}, (1 + \delta_1^+)(1 + \delta_2^+)y_A) = (4, 2, 7)\). Regarding the algorithm cost row and the \( \beta \) row (except for the \( \beta \) column) are multiplied by \( \frac{1}{\beta(\delta_2^+)} = \frac{1}{1.72} \).

The following tableau is obtained:

\[
\begin{array}{cccccccccc}
& z & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_6 & \beta & s_1 & s_2 & R_3 & R_4 & \text{RHS} \\
\hline
z & 1 & -0.07 & 0 & 0 & -0.17 & -0.64 & -0.5 & 0 & -0.14 & 0 & 0.1 & 0.25 & 1 \\
\lambda_2 & 0 & -1.5 & 1 & 0 & -0.25 & -0.25 & -0.5 & 0 & 0 & 0 & 0 & 0 & 1.75 \\
s_2 & 0 & -1.5 & 0 & 0 & -1.25 & 0.5 & 0.75 & 0 & 0 & 1 & -25 & -25 & 0 \\
\beta & 0 & -0.07 & 0 & 0 & -0.17 & -0.64 & -0.5 & 1 & -0.14 & 0 & 0.1 & 0.25 & 1 \\
\lambda_3 & 0 & -0.5 & 0 & 1 & 1.25 & 1.5 & 0.25 & 0 & 0 & 0 & 0 & 25 & -0.75 \\
\end{array}
\]

The above table is optimal for \( \beta(\delta_1^+)\beta(\delta_2^+)x_A^{ND}, x_A^P, (1 + \delta_1^+)(1 + \delta_2^+)y_A = (7, 2, 7) \).

Now, we do the parametric analysis in the direction of \( \bar{b}' = B^{-1}b' = \begin{pmatrix} 0 & 0 & -0.25 & 1.75 \\ 0 & 1 & -0.25 & -0.25 \\ -0.14 & 0 & 0.1 & 0.25 \\ 0 & 0 & -0.25 & -0.75 \end{pmatrix} \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 7 \end{array} \right) = \left( \begin{array}{c} -1.75 \\ -1.75 \\ 0.7 \\ 1.75 \end{array} \right). \]

Since \( \bar{b}'_1 < 0 \) and \( \bar{b}_1 = 0 \), then using the algorithm of the parametric analysis in linear programming, \( \lambda_2 \) leaves the basis and \( \lambda_4 \) enters the basis by a dual-simplex iteration. And hence the tableau changes to:
\[ \bar{y} = B^{-1}b' = \begin{pmatrix} 0 & 0 & 1 & -7 \\ 0 & 1 & 1 & -9 \\ -0.14 & 0 & 0.27 & 0.94 \\ 0 & 0 & -1.5 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 1.89 \\ -10.57 \end{pmatrix}. \]

Hence \( \delta_3^+ = \frac{1}{-(-10.57)} \), \( \beta(\delta) = c_B \bar{b} + \delta c_B \bar{b}' = 1 + \delta(0, 0, 1, 0) \begin{pmatrix} 7 \\ 7 \\ 1.89 \\ -10.57 \end{pmatrix} = 1 + 1.89 \delta \Rightarrow m_3^+ = 1.89 \) and \( \frac{1}{m_2^+} = 0.53. \)

Since \( \frac{1}{m_3^+} < 1 \), then the algorithm terminates and the results can be interpreted as follows:

1. If the outputs increase from 1 to \((1 + \delta_1^+)(1)=3\), then the increase rate is equal to \( \frac{1}{m_1^+} = 6.25. \)

2. If the outputs increase from 3 to \((1 + \delta_1^+)(1 + \delta_2^+)(1)=7\), then the increase rate is equal to \( \frac{1}{m_2^+} = 1.85. \)

3. \( \beta(\delta_1^+\beta(\delta_2^+)x'_D, x'_N, (1 + \delta_1^+)(1 + \delta_2^+)y_A) = (7, 2, 7) \) is an most productive scale size and increasing its outputs is not beneficial.

### 4 Conclusion

In this paper, we introduced RTS of DMUs with nondiscretionary data. The proposed model uses the parametric analysis for determining RTS of DMUs with nondiscretionary data and determines the global variation of outputs with respect to the variation of inputs.
References


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