On Some Properties of Life Distributions with Increasing Elasticity and Log-concavity

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Abstract

The purpose of this paper is to extend and systematize known results in log-concave and log-convex properties of life distributions. Also, to discuss the closure property of increasing generalized failure rate (IGFR) distributions with respect to mixing operation.

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1 Introduction

The study of the log-concavity and log-convexity are useful in many areas of economics, political science, biology, actuarial science and engineering. It is often important to make explicit assumptions on the underlying distribution. However, in some situations there is no closed form expression for the distribution functions, the failure rates, the mean residual lifetime (MRL), and the variance residual lifetime (VRL) and it is still of interest to study the properties of such functions. Most of the existing results in the literature have dealt with the density functions, distribution functions, and their integrals. Some of these results have been related to reliability functions, failure rates, and MRL functions. Yet the log-concave and log-convex properties of the VRL have not been touched. For this reason, we study the log-concavity for the VRL.

In Section 2, we give definitions and basic notations for continuous concave distributions. We also give some examples for common continuous distributions. Section 3 discusses the log-concavity involving failure rates, MRL and VRL. In Section 4, we consider log-concavity for discrete life distributions. We discuss stochastic orderings that are related to concave distributions in Section 5. Finally, we study the property of IGFR distributions.

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2 Log-concavity for continuous distributions

As usual, we assume that $X$ is a nonnegative random variable describing the lifetime of a component. The distribution function is $F(x) = P\{X \leq x\}$ and the reliability function is $F(x) = 1 - F(x)$. In some cases below, we assume that there is a density $f$, and that $f$ is a smooth function on its domain $(0, \infty)$.

**Definition 2.1** A random variable, $X$, is said to have a concave distribution if, for any $x_1, x_2$ and any $\lambda \in [0, 1]$, the following relation is satisfied for the density $f$:

$$f(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda f(x_1) + (1 - \lambda)f(x_2). \quad (1)$$

The opposite concept of concave function is for a function to be convex. Hence, $X$ is convexly distributed if the inequality sign in (1) is reversed.

**Definition 2.2** A random variable, $X$, is said to have a log-concave distribution if for any $x_1, x_2$ and any $\lambda \in [0, 1]$, we have

$$f(\lambda x_1 + (1 - \lambda)x_2) \geq f^\lambda(x_1)f^{1-\lambda}(x_2). \quad (2)$$

Assuming $f$ to be positive, we take a logarithm in (2) thus getting

$$\ln(f(\lambda x_1 + (1 - \lambda)x_2)) \geq \lambda \ln(f(x_1)) + (1 - \lambda) \ln(f(x_2)). \quad (3)$$

The function $f$ is called a log-convex function if the inequality sign in (2) is reversed. It should also be noted that if $f$ is log-concave, then it is a continuous function on its domain and, if it is differentiable, then it is continuously differentiable on its domain.

**Proposition 2.3** If $h(x), x \in D$ is a differentiable function on its domain $D, D \subset (0, \infty)$ then the following conditions are equivalent:

(a) $h(x)$ is log-concave on $D$.

(b) $h'(x)/h(x)$ is monotone decreasing in $x$.

(c) $(\ln h(x))'' < 0$.

Now, let us define the right-hand integral of the reliability function as follows: $v(x) = \int_x^\infty F(u)du$ and $V(z) = \int_z^\infty v(u)du$.

**Theorem 2.4** Suppose that $F$ is a life distribution and its density function $f$ is log-concave on $(0, \infty)$. Then the following hold:

(a) $F(x), \bar{F}(x)$ and $v(x), x > 0$, are log-concave.
On some properties of life distributions

(b) $V(x)$, $x > 0$, is log-concave.

Proof. See Theorems 1 and 3 in Bagnoli and Bergstrom (2005) for the proof of (a). Proof for claim (b) can be obtained in a similar way.

Remark 2.5 Let us mention that log-concavity of the density function is also known as the total positive property, denoted by $TP_2$. This property was introduced and discussed by Barlow and Proschan (1981).

We state the following theorem, which can be considered as dual to Theorem 2.4, so we do not need details.

Theorem 2.6 Suppose that $F$ is a life distribution and the density function $f$ is log-convex on $(0, \infty)$. Then the following hold:

(a) $F(x)$, $\bar{F}(x)$ and $v(x)$, $x > 0$, are log-convex.
(b) $V(x)$, $x > 0$, is log-convex.

Some continuous distributions are log-concave on their domains and some are log-convex. There are also distributions which are log-concave, or log-convex at certain intervals and at others are neither log-concave nor log-convex. Table 2 gives details of such distributions.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Domain</th>
<th>Density function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential: $Exp(\lambda)$</td>
<td>$x \in \mathbb{R}^+$</td>
<td>Log-concave and log-convex</td>
</tr>
<tr>
<td>Uniform: $Un(a, b)$</td>
<td>$x \in [a, b]$ where $a, b \in \mathbb{R}$</td>
<td>Log-concave</td>
</tr>
<tr>
<td>Normal: $N(\mu, \sigma^2)$</td>
<td>$x \in \mathbb{R}^+$, $\mu \in \mathbb{R}$, $\sigma &gt; 0$</td>
<td>Log-concave</td>
</tr>
<tr>
<td>Gamma: $\gamma(\theta, \beta)$</td>
<td>$x \in \mathbb{R}^+$ and $\theta \geq 1$</td>
<td>Log-concave</td>
</tr>
<tr>
<td>Beta: $B(a, b)$</td>
<td>$x \in \mathbb{R}^+$ and $a, b \geq 1$</td>
<td>Log-concave</td>
</tr>
<tr>
<td>Weibull: $Wei(\gamma, \theta)$</td>
<td>$x \in \mathbb{R}^+$ and $\theta \geq 1$</td>
<td>Log-concave</td>
</tr>
<tr>
<td>Pareto: $Par(\theta)$</td>
<td>$x \in \mathbb{R}^+$ and $\theta &gt; 0$</td>
<td>Log-convex</td>
</tr>
<tr>
<td>Log-normal $LN(\mu, \sigma^2)$</td>
<td>$x \in \mathbb{R}^+$</td>
<td>Neither log-concave nor log-convex</td>
</tr>
</tbody>
</table>

Table 1: Log-concave and log-convex continuous distributions

3 Log-concavity for some ageing classes

Here we discuss log-concavity for the main characteristics: failure rate, MRL and VRL. We also discuss log-concavity for the reversed failure rate, reversed mean residual (RMR) and reversed variance residual (RVR).
We briefly recall the definitions of the failure rate function $r(x)$, $x \geq 0$ and the reversed failure rate function $\tilde{r}(x)$, $x \geq 0$ corresponding to a lifetime $X$, where $X \sim F = \{F(x), x \geq 0\}$:

$$r(x) = -\frac{d}{dx} \ln \bar{F}(x) = f(x)/\bar{F}(x) \quad \text{and} \quad \tilde{r}(x) = \frac{d}{dx} \ln F(x) = f(x)/F(x). \quad (4)$$

The following proposition is well-known in the literature, see e.g. Chandra and Roy (2001). Details are omitted.

**Proposition 3.1** If the density function $f(x)$, $x \geq 0$ is log-concave, then the following hold:

(a) The function $r(x)$, $x \geq 0$ is monotone increasing and hence, $F \in IFR$.

(b) The function $\tilde{r}(x)$, $x \geq 0$ is monotone decreasing and hence, $F \in DRFR$.

The log-concavity property was defined in terms of the distribution function and the density function. It is of interest to see whether or not such a property holds for the mean residual and the reversed mean residual lifetime.

Let us recall that if $X \sim F = \{F(x), x \geq 0\}$, then there is another random variable associated with $X$ called the residual random variable $X_t = [X - t | X \geq t]$, where $t \geq 0$. The expectation of $X_t$ is known as the mean residual lifetime $\mu(t)$, where

$$\mu(t) = \mathbb{E}[X - t | X \geq t] = \frac{1}{F(t)} \int_t^\infty \bar{F}(u)du, \quad t \geq 0. \quad (5)$$

It is reasonable to expect in real situations to find random variables which are not related only to the future, but can also refer to the past. One possibility is to consider the random variable $\tilde{X}_t = [t - X | X \leq t]$ as representing the reversed time, or the inactivity time. The expectation of $\tilde{X}_t$ is called the reversed mean residual, or the mean inactivity $\tilde{\mu}(x)$, where

$$\tilde{\mu}(t) = \mathbb{E}[t - X | X \leq t] = \frac{1}{F(t)} \int_0^t F(u)du, \quad x \geq 0 \quad (6)$$

The function $\mu(x)$, $x \geq 0$ can have different shapes. One possibility is to be decreasing which is the natural shape in many real situations.

**Theorem 3.2** The mean residual life function $\mu(x)$, $x \geq 0$, is monotone decreasing if and only if the function $v(x) = \int_x^\infty \bar{F}(u)du$ is log-concave.

**Proof.** Notice that $v'(x)/v(x) = (\ln v(x))'$ and also that the function $\mu(x)$ can be written in terms of $v(x)$ as follows:

$$\mu(x) = -v(x)/v'(x), \quad x \geq 0. \quad (7)$$
Then it is not difficult to get the following:

\[
\ln v(x) = -1/\mu(x), \quad x \geq 0, \quad \text{and}
\]

\[
\ln v(x)'' = \mu'(x)/\mu^2(x) \leq 0, \quad \text{(since } \mu'(x) \leq 0).}
\]

Hence the function \( v(x), x > 0, \) is log-concave. The ‘only if’ part is obvious.

**Theorem 3.3** The reversed mean residual \( \tilde{\mu}(x), x > 0, \) is monotone increasing if and only if \( \tilde{\nu}(x) = \frac{1}{\ln \tilde{v}(x)}, x > 0 \) is log-concave.

**Proof.** Again, by noticing that the function \( \tilde{\mu}(x) \) can be expressed in terms of \( \tilde{\nu}(x) \) as \( \tilde{\mu}(x) = \frac{1}{\ln \tilde{v}(x)} \), the result immediately follows.

We continue the analysis of log-concavity, now for the variance residual lifetime function \( \sigma^2(x), x \geq 0. \) We also consider the reversed variance residual lifetime \( \tilde{\sigma}^2(x), x \geq 0. \)

By definition, \( \sigma^2(x) = \mathbb{V}[X - x|X \geq x] = \mu_2(x) - \mu^2(x), \) where \( \mu_2(x) \) is the second conditional moment and \( \mu(x) \) is the first conditional moment of \( X. \)

Here, we assume that the second moment of \( X \) is finite. We have:

\[
\sigma^2(x) = \frac{2}{F(x)} \int_x^\infty \int_y^\infty \bar{F}(u)du dy - \mu^2(x), \quad x \geq 0.
\]  \( \tag{8} \)

Before discussing the log-concavity for the function \( \sigma^2(x) \), we will study this property for the equilibrium mean residual lifetime \( \mu_e(x), x \geq 0. \)

**Theorem 3.4** The equilibrium mean residual lifetime \( \mu_e(x), x \geq 0 \) is monotone decreasing if and only if the function \( V(x), x \geq 0 \) is log-concave.

**Proof.** By definition we have \( V(x) = \int_x^\infty v(u)du, \) and \( v(y) = - \int_y^\infty \bar{F}(u)du, \) hence we can write that \( V'(x) = -v(x), x \geq 0 \) and then, that

\[
\mu_e(x) = \frac{1}{v(x)} \int_x^\infty \int_y^\infty \bar{F}(u)du dy = -V(x)/V'(x).
\]  \( \tag{9} \)

Thus, we obtain the following chain of relations:

\[
\mu_e(x) \text{ is decreasing } \iff \quad V(x)V''(x) - (V'(x))^2 \leq 0,
\]

\[
\iff \quad (V'(x)/V(x))' \leq 0,
\]

\[
\iff \quad V(x) \text{ is log-concave.}
\]

The result follows.

The next theorem is for the variance residual lifetime function \( \sigma^2(x), x \geq 0. \)

**Theorem 3.5** The variance residual lifetime function \( \sigma^2(x), x \geq 0 \) is monotone decreasing if and only if the function \( V(x), x \geq 0 \) is log-concave.
Proof. We recall that if $F$ is a life distribution with a finite second moment, then $F$ has a decreasing variance residual lifetime (DVRL) property, or $F \in \text{DVRL}$ if and only if

$$\sigma^2(x) \leq \mu^2(x), \quad x \geq 0. \quad (10)$$

Assuming that $F \in \text{DVRL}$ and noting that $V'(x) = -v(x)$ and $V''(x) = \bar{F}(x)$, we obtain the following chain of relations:

$$\sigma^2(x) \leq \mu^2(x) \iff \bar{F}(x)V(x) \leq v^2(x),$$
$$\iff V(x)V''(x) \leq (V'(x))^2,$$
$$\iff V(x)V''(x) - (V'(x))^2 \leq 0,$$
$$\iff (V'(x)/V(x))^t \leq 0,$$
$$\iff V(x) \text{ is log-concave.}$$

The result follows.

Remark 3.6 The dual classes of life distributions with decreasing equilibrium mean residual lifetime and decreasing variance residual lifetime are characterized by involving the property log-convex instead of log-concave.

4 Log-concavity for discrete distributions

In previous sections, we have studied log-concavity properties for continuous ageing classes. Now, we deal with discrete lifetime systems. In reliability analysis, discrete life distributions are used for modelling systems when measurements are taken discretely. Let $X \sim (\mathcal{P}, N)$ with values in the set $N = \{0, 1, 2, \cdots\}$ and probability mass function $\mathcal{P} = \{p_k, k \in N\}$, where $p_k = \mathbb{P}\{X = k\}$.

Definition 4.1 A discrete random variable $X$, (or $\mathcal{P}$) is said to be log-concave if

$$p^2_{k+1} \geq p_k p_{k+2} \quad \text{for all } k \in N. \quad (11)$$

Equivalent to the sequence $\{p_{k+1}/p_k\}$ is decreasing for all $k \in N$.

Gupta et al. (1997) define Glaser’s function (also known as eta-function) for a discrete random variable $X$ as

$$\eta_k = 1 - \frac{p_{k+1}}{p_k}, \quad k \in N. \quad (12)$$
They show that if a discrete distribution is log-concave, then it has discrete increasing failure rate (D-IFR), and if it is log-convex, then it has discrete decreasing failure rate (D-DFR).

The following theorem discusses the log-concavity of the discrete mean residual life function $L_k$, $L_k = \mathbb{E}[X - k | X > k]$, or

$$L_k = \frac{1}{B_{k+1}} \sum_{i=k+1}^{\infty} B_i, \quad k \in \mathbb{N}; B_i = p_i + p_{i+1} + \ldots$$ (13)

Define the right-hand summation $v_k = \sum_{i=k+1}^{\infty} B_i, \quad k \in \mathbb{N}$.

**Theorem 4.2** The discrete mean residual life function $L_k$, $k \in \mathbb{N}$ is monotone decreasing if and only if the function $v_k$, $k \in \mathbb{N}$ is log-concave.

**Proof.** We first note that $B_{k+1} = v_{k+1} - v_k$, for all $k \in \mathbb{N}$ and by definition we have:

$v_k$ is log-concave $\iff v_{k+1}^2 \geq v_k v_{k+2}$,
$\iff v_{k+1} (v_{k+1} - v_k) \geq v_k (v_{k+2} - v_{k+1})$,
$\iff v_{k+1} B_{k+1} \leq v_k B_{k+2}$,
$\iff L_{k+1} \leq L_k, \quad k \in \mathbb{N}$.

Thus, $L_k$, $k = 0, 1, 2, \ldots$, is monotone decreasing.

Note that such a property for the dual class D-IMRL can be derived and this holds if and only if the function $v_k$, $k \in \mathbb{N}$ is log-convex.

**Remark 4.3** As in the continuous case, several discrete life distributions are log-concave. For examples, the binomial distribution, Poisson distribution and negative binomial distribution obey this property.

5 Stochastic orderings

When studying random variables, the set of possible values and the way of concentration of the probability mass are both important. There is a big difference between two variables, $X$ and $Y$, with the same set (interval) of values, if e.g. most of the mass for $X$ is on ‘smaller’ values, then most of the mass for $Y$ is on ‘larger’ values.

Suppose further that $X$ and $Y$ are two random variables that both have equal mean and describe the utilities of two risky investments. Then a decision maker will choose the investment with lower variability, measured, for example by a lower variance. Therefore variability orderings are of great interest in the context of decision making under risk.
Definition 5.1 Let \( X \) and \( Y \) be two random variables with finite means. Then we say that:

(a) \( X \) is smaller than \( Y \) in the sense of the concave order (denoted by \( X \leq_{CV} Y \)), if for any concave function \( f \), \( \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)] \).

(b) \( X \) is smaller than \( Y \) in the increasing concave order (denoted by \( X \leq_{ICV} Y \)), if for any increasing concave function \( f \), \( \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)] \).

(c) \( X \) is smaller than \( Y \) in the sense of the convex order (denoted by \( X \leq_{CX} Y \)), if for any convex function \( f \), \( \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)] \).

(d) \( X \) is smaller than \( Y \) in the increasing convex order (denoted by \( X \leq_{ICX} Y \)), if for any increasing convex function \( f \), \( \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)] \).

An equivalent form of property (b) can be formulated as follows: If \( \bar{F} = 1 - F \) and \( \bar{G} = 1 - G \) are the reliability functions associated with \( X \) and \( Y \), then

\[
X \leq_{ICV} Y \iff \int_0^t \bar{F}(u)du \leq \int_0^t \bar{G}(u)du, \quad \text{for all } t \geq 0. \tag{14}
\]

Also property (d) is equivalent to

\[
X \leq_{ICX} Y \iff \int_t^\infty \bar{F}(u)du \geq \int_t^\infty \bar{G}(u)du, \quad \text{for all } t \geq 0. \tag{15}
\]

These stochastic orders are used in the literature under various other names, see e.g., Ross (1983), who called the order \( \leq_{ICX} \) ‘stochastically more variable’, while Müller and Stoyan (2002) called it ‘smaller in mean residual life’. It is known as ‘stop-loss order’ in the actuarial science literature, and as ‘second order stochastic dominance of second type’ denoted by \( \leq_{SSD_2} \) in economics.

The stochastic orders can be characterized by the sign of the first derivatives or the first and the second derivatives. We know from the classical analysis that a differentiable function \( f \) is increasing if the first derivative is nonnegative, and it is convex if the second derivative is nonnegative. This idea and some extensions have been considered, e.g. by Rolski and Stoyan (1974) with applications in queuing theory, and by Denuit (2001) in actuarial science.

Definition 5.2 Let \( X \) and \( Y \) be two random variables with finite means and let \( \mathbb{E}[X - t]_+ = \int_t^\infty \bar{F}(u)du \). Then:

(a) \( X \) is said to be smaller than \( Y \) with respect to the \( s \)-increasing-concave order (denoted by \( X \leq_{s-ICV} Y \)), if \( \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)] \) for all \( s \)-increasing-concave function \( f \). Equivalently: \( \mathbb{E}[X - t]_+^{s-1} \leq \mathbb{E}[Y - t]_+^{s-1}, \) for all \( t \geq 0 \) and \( s = 2, 3, \ldots \).
(b) $X$ is said to be smaller than $Y$ with respect to the $s$-increasing-convex order (denoted by $X \leq_{s-ICX} Y$), if $E[f(X)] \leq E[f(Y)]$ for all $s$-increasing-convex function $f$. Equivalently: $E[X - t]_{+}^{s-1} \geq E[Y - t]_{+}^{s-1}$, for all $t \geq 0$.

Remark 5.3 With increasing $s$, both orderings $\leq_{s-ICX}$ and $\leq_{s-ICV}$ become weaker. In the order $\leq_{s-ICV}$, if $s \to \infty$, then we get the so-called Laplace transform order (denoted by $\leq_{Lt}$).

Shaked (1994) observes that the mean residual life order can be characterized by the increasing convex order as follows:

\[ X \leq_{MRL} Y \implies X \leq_{ICX} Y \tag{16} \]

This means that the order $\leq_{MRL}$ is more informative than the order $\leq_{ICX}$.

Another important characterization can be established using the residual random variables $X_t$ and $Y_t$:

\[ X \leq_{MRL} Y \iff X_t \leq_{ICX} Y_t, \quad \text{for all } t \geq 0. \tag{17} \]

Ahmad et al. (2005) follow the same approach and give characterization for the reversed mean residual life order as follows:

\[ X \leq_{RMR} Y \implies X \leq_{ICV} Y, \quad \text{for all } t \geq 0. \tag{18} \]

\[ X \leq_{RMR} Y \implies X_t \leq_{ICV} Y_t, \quad \text{for all } t \geq 0. \tag{19} \]

Definition 5.4 A random variable $X$, or its distribution function $F$, $X \sim F$, is said to be smaller than $Y \sim G$ in variance residual lifetime ordering (denoted by $X \leq_{VRL} Y$) if the following ratio

\[ \frac{\int_{x}^{\infty} \int_{t}^{\infty} F(u) dudt}{\int_{x}^{\infty} \int_{t}^{\infty} G(u) dudt}, \quad x \geq 0, \quad \text{is nondecreasing.} \tag{20} \]

Next, we characterize the increasing convex and concave orders that are related to the variance residual lifetime and reversed variance residual lifetime.

Theorem 5.5 Suppose that $X$ and $Y$ are two random variables with finite second moments. Then

\[ X \leq_{VRL} Y \iff X_t \leq_{3-ICX} Y_t, \quad \text{for all } t \geq 0. \tag{21} \]

Proof. The proof is straightforward, once we note the following:

\[ X \leq_{3-ICX} Y \iff \int_{x}^{\infty} \int_{t}^{\infty} F(u) dudt \geq \int_{y}^{\infty} \int_{t}^{\infty} G(u) dudt, \]

and the ‘if and only if’ relation between the variance residual lifetime ordering and the equilibrium mean residual lifetime ordering:

\[ X \leq_{VRL} Y \iff X \leq_{EMRL} Y. \]
Definition 5.6 A random variable $X$ is said to be smaller than $Y$ in the sense of the reversed variance residual lifetime (denoted as $X \leq_{RVR} Y$) if, for all $x \geq 0$

$$\frac{\int_0^x \int_0^t \bar{F}(u) du \, dt}{\int_0^x \int_0^t G(u) du \, dt}$$

is nonincreasing. \hfill (22)

Theorem 5.7 Suppose that $X$ and $Y$ are two random variables with finite second moments. Then

$$X \leq_{RVR} Y \implies X_t \leq_{3-ICV} Y_t.$$ \hfill (23)

Proof. The proof is straightforward and therefore omitted.

6 Life distributions with increasing elasticity

The concept of increasing generalized failure rate (IGFR) was found to be useful in supply chain models, and also in stochastic models of service systems, see e.g. Lariviere and Porteus (2001), Anand (2005) and Ziya et al. (2004).

As usual, we let $X$ be a random variable representing the lifetime of a component and denote by $F$ its distribution; $\bar{F} = 1 - F$. We further assume that $F$ has a density function $f$. The failure rate of $X$ (or $F$) is $r(x) = f(x)/\bar{F}(x)$, $x \geq 0$. We recall that $X$ has an increasing failure rate or, equivalently, $F \in IFR$ if the function $r(x)$, $x \geq 0$ is increasing.

Lariviere and Porteus (2001) define the generalized failure rate of $X$ as follows:

$$R(x) = x r(x), \quad x > 0.$$ \hfill (24)

Definition 6.1 A life distribution $F$ has an increasing generalized failure rate, and we write $F \in IGFR$, if the function $R(x)$, $x > 0$ is increasing.

Decreasing generalized failure rate (DGFR) distributions can be defined analogously. Clearly, if $X$ is IFR, then it is also IGFR. The converse is not generally true; there are many DFR distributions which are IGFR.

In economics, for example, if $x$ is a price of a commodity and $\bar{F}(x)$ denotes the demand on that commodity, then the “elasticity” of the demand is defined by

$$e(x) = -x \bar{F}'(x)/\bar{F}(x), \quad x \geq 0.$$ 

Lariviere and Porteus (2001) adopt this concept and apply it to the supply chain management. Lariviere (2006) derives characterization for IGFR distributions in terms of IFR properties. In what follows, we discuss the closure of IGFR with respect to the mixing operation.
6.1 Closure Property

Closure properties of IGFR were considered by Anand (2005). In particular, the convolution and shifting properties are not preserved. In this section, we focus our attention on mixtures of IGFR distributions. We discuss the following question: Are the IGFR distributions closed under mixing? This will take us back to the notions and definitions of mixture distributions.

Let $X_1 \sim F_1$ and $X_2 \sim F_2$ be random variables representing lifetimes of two independent components. For any $p \in [0, 1]$, we define the functions

$$H = pF_1 + (1 - p)F_2, \quad \text{and} \quad \bar{H} = p\bar{F}_1 + (1 - p)\bar{F}_2.$$  

(25)

It is easy to see that $H$ is a “new” life distribution with $\bar{H}$ being its reliability (survival) function. Recall that $H$ is called a $p$-mixture of $\{F_1, F_2\}$. We can easily extend this definition to a larger set of life distributions. This can also be expressed in terms of densities, assuming they exist.

Suppose now that $R_1(x), x > 0$ and $R_2(x), x > 0$, are the generalized failure rate functions associated with $X_1$ and $X_2$, respectively. This means that $R_1(x) = x r_1(x)$ and $R_2(x) = x r_2(x)$, where $r_1(x)$ and $r_2(x), x > 0$ are the usual failure rates of $F_1$ and $F_2$. Then the generalized failure rate function $R(x)$, based on $R_1(x)$ and $R_2(x), x > 0$, is defined as follows:

$$R(x) = a(x) R_1(x) + (1 - a(x)) R_2(x), \quad x > 0,$$  

(26)

with a mixing function $a(x), x > 0$, where

$$a(x) = \frac{p\bar{F}_1(x)}{p\bar{F}_1(x) + (1 - p)\bar{F}_2(x)}.$$  

The next theorem tells us whether or not IGFR property is preserved under mixing operation.

**Theorem 6.2** Suppose that $F_1$ and $F_2$ are IGFR. Then their mixture $H$ is not necessarily IGFR.

**Proof.** Consider two lifetimes, $X_1 \sim Exp(4)$ and $X_2 \sim \gamma(1, 1)$. Their densities are: $f_1(x) = 4e^{-4x}$ and $f_2(x) = xe^{-x}$. Then the generalized failure rate functions are:

$$R_1(x) = 4x \quad \text{and} \quad R_2(x) = \frac{x^2}{x + 1}, \quad x > 0.$$  

(27)

It is easy to check that $R_1(x), x > 0$ and $R_2(x), x > 0$ are both increasing functions converging to $\infty$ as $x \to \infty$. Hence, $F_1 \in$ IGFR and $F_2 \in$ IGFR. Consider
now the $p$-mixture of $F_1$ and $F_2$ and its generalized failure rate function. We first chose $p = 0.95$ and easily find the function $a(x)$, namely,

$$a(x) = \frac{0.95 e^{-4x}}{0.95 e^{-4x} + 0.05 e^{-x}(x + 1)}.$$  

(28)

Hence, the generalized failure rate function is

$$R(x) = \frac{x(x + 76 e^{-3x})}{19 e^{-3x} + x + 1}.$$  

(29)

It is clear that $R(x), x > 0$ has the U property, i.e. it has an upside-down bathtub shape as shown in Figure 1.

Figure 1: Example of a mixture of two increasing generalized failure rate yielding an upside-down bathtub shape
References


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